**Absolute Relativity / Overall V2 Theory – v0.9**  
Document: (3) V1 – Formal Framework  
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Project tokens: **“Absolute Relativity (AR)”** – ETH contract 0xAacCd7bA616405C184335F193fEf080fC982921F, SOL mint ARafKuCqRgszXZWjYGWyBT7GnLZkyiaXQd1YjXC1x224  
Project wallets (on-chain records): ETH 0x1F06ea3554aE665e713a637eD136a5065C9cD787, SOL 7mik22AsVKX2ueqSWHCD8HBMpcfEMhbKUb85xYoaxCKN  
For full project, token, and on-chain record details, see **(0) Front Matter & File Map.docx, §0.4**.

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**0. Front Matter & Orientation**

**0.1 Abstract**

This volume records the core mathematical and conceptual structure of the first version of Absolute Relativity (AR), stripped down to the theory and its internal algebra. The starting point is a qualia-first, relational ontology: reality is modeled as a sequence of discrete “ticks” of experience, each carrying a Present-Moment Sphere (PMS) that splits into an Inner Network (IN, committed record) and an Outer Network (ON, admissible potential). On this PMS we define a small set of primitive operators acting on abstract tick-states—Renew, Sink, Trade, Sync, and a Framing composite—together with a ledger that tracks record and exposure budgets. Non-commutation among these operators and monotonicity properties of the ledger generate an intrinsic arrow of time and support the construction of a Lorentz-like invariant interval directly from flip counts, without presupposing a spacetime background.

The IN side of each PMS is modeled as the attractor of an iterated-function system (IFS) with a fractal dimension confined to a fixed range, and we introduce a pivot function that controls how strongly different contexts couple. A dual-exponent description of record versus potential thickening yields an effective present-moment dimension close to two and motivates treating the PMS boundary as a distinguished two-dimensional “hinge.” A complex structure on a two-dimensional present plane provides the minimal phase-like machinery needed to support Born-style probability weights, while a context-gate rule based on distinguishes full-strength interactions within a Collective Sphere (CS) from attenuated interactions across context boundaries. Above this, the theory introduces a discrete context ladder indexed by integers , with an effective dimension curve and corresponding pivot weights . A reproduction kernel defines a memory fractal dimension , and a master action on the ladder—weighted by the pivot factors—admits a continuum embedding that yields field-like equations and a canonical quantization scheme. Quantization is expressed as a path-sum over flip histories in context-time, with CS-sampling and IN-measures recovering standard quantum-mechanical postulates (superposition, Born rule) as structural consequences of the flip algebra rather than independent axioms.

On this ladder we build formal gauge, matter, and gravitational sectors: U(1) and non-Abelian gauge structures arise from context connections and pivot-weighted actions, matter spectra are encoded as eigenstructures of ladder and projector operators, and an effective gravitational sector appears from the hinge and pivot profile, including inverse-square behavior and horizon formation in purely relational terms. A final layer singles out the zero-context band with as a true pivot: collapse kernels on the 2-sphere boundary become identity at this band, mirror-symmetry functionals on locate it as the unique center, and an extension to null-cone shells yields a retarded composite operator that also reduces to identity at the pivot. From this, special-relativistic kinematics, a unified inverse-square field, and a compact four-dimensional scalar field equation (with both gravitational and Gauss-law-type limits) are derived as direct consequences of context flips and the hinge geometry.

Throughout, this volume is purely theoretical: it defines primitives, operators, dimension curves, pivot functions, actions, and structural theorems about their behavior. In addition, it specifies an abstract present-act engine (Section 12) compatible with this algebra—state space, gate predicates, acceptance rule, typed budgets, and hinge constants—and expresses gravity as a feasibility geometry inside that engine, all at a symbolic level. It does *not* contain simulation designs, numerical fits, or empirical datasets, and it does not present any implementation-level engine details, code, or calibration procedures. Philosophical exposition and empirical probes (matter-addition and lensing simulations, context-level surveys, falsifiability discussions) are treated in separate companion documents; here they appear only as external pointers where needed to note that particular structures are testable. The intent is for this text to serve as the canonical, simulation-independent record of the AR V1 model as a formal framework.

**0.2 Scope & Non-Goals**

This volume is intended to be the **canonical record of the V1 Absolute Relativity (AR) formalism**, independent of any particular implementation, dataset, or philosophical interpretation. It collects, in one place, the **definitions, operators, algebraic laws, geometric structures, and field constructions** that make up the theory, and it presents them in a way that is internally self-contained.

**0.2.1 Scope**

Within this volume we:

**Define the primitive ontology and present-moment objects**

* Ticks as atomic present-moment updates.
* Present-Moment Spheres (PMS) with Inner (IN) and Outer (ON) networks.
* Collective Spheres (CS) as synchronized collections of PMSs that act as frames.
* An informal context ladder indexed by integers (inner, hinge, outer bands).

**Specify the tick-operator algebra, ledger, and invariant interval**

* The primitive operators (Renew, Sink, Trade, Sync, Framing) acting on tick-state carriers .
* Ledger functions with , their monotonicity properties, and the induced intrinsic arrow of time.
* Flip words, flip-count vectors , and the construction of interval components satisfying a Lorentz-like invariant quadratic form

together with CS-based frame and frame-transformation structure.

**Introduce the fractal inner geometry, pivot function, and present plane**

* IN as an iterated-function-system (IFS) attractor with effective fractal dimension in a bounded range.
* The pivot function , positive and normalized by , with hinge-centered constraints and cross-scale regularity.
* Dual time exponents , and the present-moment exponent .
* A 2-dimensional present plane with complex structure , present amplitudes , and a structural Born-style rule tying to IN basins and their reference measure.

**Construct the unified fractal–sink paradigm and the context ladder**

* A radial dimension profile with a hinge at , and a boundary projector that collapses radial structure to the hinge.
* A discrete context ladder with bands , boundary graphs , collapse and expansion operators , and a reproduction kernel with associated memory dimension .
* A logistic-type discrete dimension curve consistent with the radial profile, together with bandwise pivot weights and hinge area-law behavior at .

**Define the master action, context-time continuum, quantization, and RG structure**

* A discrete and continuum master action on the ladder, with band contributions weighted by .
* Context-time Hamiltonian structure, canonical momenta, and Euler–Lagrange equations for band fields.
* Canonical quantization in context-time, path-sum representations over ladder histories, and CS sampling leading back to the Born-style rule at the hinge.
* A renormalization-group (RG) interpretation of context flips, flows of , and other couplings, and inner/hinge/outer phases of the ladder with built-in UV/IR behavior.

**Build formal gauge, matter, and gravitational sectors**

* U(1) and non-Abelian (e.g. SU(2), SU(3)) gauge structures on the boundary graphs , with link variables, covariant differences, Wilson loops, and pivot-weighted Yang–Mills–type actions.
* Matter spectra as eigenstructures of ladder + gauge operators, defined via context eigenstates and projectors, with mass-like labels and band-support profiles.
* A gravitational sector built from the hinge and pivot profile, including area-law horizons, inverse-square behavior, and ladder-level field equations in a purely relational, context-based form.

**Single out the pivot band and unified 4D structures**

* Collapse kernels on the hinge boundary (S²-like) and null-cone shells, mirror-symmetry and pivot-location functionals on , and retarded composite operators that reduce to identity at the pivot.
* A unified inverse-square field and compact 4D scalar field equation obtained via hinge projection and hinge “thickening”, with Maxwell- and Poisson-type limits as special cases.

**Specify an abstract present-act engine and feasibility geometry**

* An abstract V2.1 present-act engine (Section 12), given at the structural level: tick-local state space, feature alphabet and gates, ratio-lexicographic acceptance rule, typed budgets , and hinge constants consistent with the earlier algebra and interval.
* Gravity as a feasibility geometry in this engine: a unit-free gravity amplitude built from shell-wise survival fractions under a radial ParentGate, nested-sphere structures, and horizon templates, all expressed symbolically and without numeric calibration.

Every core definition, axiom, proposition, and theorem of the V1 AR formalism either appears explicitly in this volume or is referenced through the indices and cross-reference maps in Part XIII, so that the theory can be read as a self-contained mathematical framework independent of particular simulations, datasets, or philosophical narratives.

### **0.2.2 Non‑Goals**

Equally important is what this volume does not attempt to do:

**No empirical datasets or numerical fits**

There are no CL probes, no survey names, no simulation outputs, no parameter fits, and no plots. Any statement that a structure is testable is limited to a brief remark; all concrete tests, datasets, and results are deferred to separate Evidence/Simulation volumes.

**No implementation‑level engine design or applications**

Where this volume introduces the V2.1 present‑act engine (Section 12), the treatment is strictly structural: we specify its abstract state space, gate predicates, acceptance rule, typed budgets, and hinge constants as part of the core AR V1 algebra. We do **not** present any concrete data structures, numerical schemes, code, pipelines, or scene/mesh designs for running the engine in practice. Detailed engine implementations, gravity simulations built on ParentGate, and application‑specific configurations belong to dedicated engine and evidence documents.

**No extended philosophical exposition**

This volume does not argue for a qualia‑first ontology, does not survey competing metaphysical views, and does not develop the broader philosophical story of AR. The qualia‑first premise appears only as a minimal statement sufficient to name the primitives; full philosophical treatment is left to a separate interpretive volume.

**No phenomenological catalog or model‑to‑data matching**

We do not derive or compare specific particle masses, mixing angles, cosmological constants, or astrophysical profiles. References to familiar physical theories (relativity, quantum theory, gauge theories, gravity) are strictly structural: the goal is to show how similar equations arise from the AR framework, not to present a complete phenomenology.

**No experimental roadmap or falsifiability discussion**

We do not discuss experimental design, statistical methods, or criteria for acceptance/rejection of AR. Those topics, together with detailed proposals for tests, are confined to companion documents.

In short, this volume should be readable as if it were a stand‑alone mathematical theory: it tells the reader what the objects are, what operations act on them, what structures emerge from those operations, and how these structures fit together into a coherent framework. All questions of implementation, measurement, interpretation, and empirical performance are intentionally kept outside its scope.

**0.3 Notation & Conventions**

This section fixes the basic symbols and formatting conventions used throughout the volume. The goal is to keep the alphabet small and consistent so the same object is always written the same way in every layer of the theory.

**0.3.1 Indices, sets, and basic symbols**

* **Indices**
  + : tick index (discrete “time step” in the sequence of present-moment updates).
  + : context index on the discrete context ladder (…, ).
  + or : radial/context variable used for continuous versions of the ladder when convenient.
  + : generic integer indices (e.g. for eigenmodes, components, or summations).
* **Standard sets**
  + : non-negative integers.
  + : all integers.
  + : real numbers.
  + : complex numbers.
* **Indicator and projection notation**
  + : indicator of a set ; equals 1 if , 0 otherwise.
  + : projection map onto a set or subspace .

**0.3.2 Present-moment objects and networks**

* **Ticks and carriers**
  + A **tick** is labeled by an integer .
  + A **carrier** at tick is written

where is the abstract state at tick , is the inner network, and is the outer network.

* **Present-Moment Sphere (PMS)**
  + The **Present-Moment Sphere** at tick is denoted

viewed as the boundary between record and potential at that tick.

* **Inner and outer networks**
  + : the inner network (committed “record” structure at tick ).
  + : the outer network (admissible “potential” structure at tick ).
* **Collective Sphere (CS)**
  + A **Collective Sphere** is written when context is clear, or when multiple CSs are present, and denotes a synchronized collection of PMSs that share a common effective boundary configuration.

**0.3.3 Operator notation and flip words**

* **Primitive operators**
  + : Renew operator.
  + : Sink operator.
  + : Trade operator.
  + : Sync (Collective Sphere) operator.
  + : Framing operator, understood as the composite when the composition is well-defined.
* **Domains and action**
  + Each operator acts on carriers:

where is the domain of admissible carriers for .

* + When it is useful to emphasize the “hat” operator viewpoint on an abstract state space , we write acting on , but for most of the text the plain letters are used.
* **Operator words and flip counts**
  + A **flip word** (or operator word) is a finite composition

with each .

* + The **flip-count vector** associated with (relative to a chosen carrier) is denoted

where each component counts how many times the corresponding primitive appears in .

* **Neutral moves**
  + A flip word is called **neutral** if its flip-count vector vanishes, ; such compositions are treated as descriptive redundancy and do not change invariants that depend only on .

**0.3.4 Ledger, budgets, and invariants**

* **Ledger and budgets**
  + : inner/record budget (non-negative real or integer).
  + : exposure/potential budget.
  + : capacity, defined so that

often with assumed locally constant along admissible evolutions.

* **Interval components**
  + From a flip-count vector we define three quantities:
    - : coordinate-time increment.
    - : proper-time increment.
    - : spatial separation magnitude.
  + These are combined into a Lorentz-like quadratic invariant

where is the characteristic conversion constant between spatial and temporal units.

* **Frames and transforms**
  + Transformations that map carriers between frames while preserving the invariant interval form a group denoted ; explicit matrix notation is introduced only when needed.

**0.3.5 Dimensions, pivot function, and ladder indices**

* **Fractal dimension**
  + : effective fractal dimension of the IN attractor associated with a given context or PMS boundary, typically constrained to a fixed interval (e.g. ).
* **Pivot function**
  + : dimension-dependent pivot function, a positive scalar weight with normalization . It is evaluated either at local dimensions or at ladder values .
* **Dimension curve on the ladder**
  + : effective dimension assigned to context band .
  + : deviation from the hinge dimension, with
* **Memory dimension**
  + : memory fractal dimension associated with reproduction across context band .
* **Radial profile**
  + When a continuous radial variable is used, we write for the dimension profile across radial contexts and keep the hinge condition .

**0.3.6 Claims, labels, and references within the volume**

* **Labeling of formal statements**
  + **Definitions** are labeled “Definition X.Y.Z” and may be abbreviated as “DEF-ID” in the indices.
  + **Axioms** are labeled “Axiom X.Y.Z” and may be abbreviated as “AX-ID”.
  + **Propositions** and **Theorems** are labeled “Proposition X.Y.Z” or “Theorem X.Y.Z” and may be abbreviated as “PROP-ID” or “THM-ID”.
  + Only these labeled statements are treated as core formal claims of the theory.
* **Cross-references**
  + Internal references use section numbering, e.g. “see Section 4.3” or “see Definition 2.2.1”.
  + References to simulations, datasets, or philosophical discussions are limited to neutral phrases such as “see Evidence Volume” or “see Interpretive Volume” without further detail in this text.

**0.3.7 Units, constants, and stylistic conventions**

* **Units and constants**
  + The constant (often set to 1 in intermediate algebra) is the unique scale factor that appears in the invariant interval. When set explicitly, it is treated as a positive real constant.
  + No other physical constants are fixed numerically in this volume; any appearance of characteristic scales is symbolic only.
* **Stylistic conventions**
  + Scalars are written in plain italic (e.g. ).
  + Vectors are written with an arrow or boldface when needed (e.g. or ), but for most of the theory only magnitudes appear.
  + Operators on state spaces are denoted by capital letters , with optional hats when clarification is useful.
  + Context indices are always integers , and radial variables are reserved for continuous analogues.

These conventions apply globally; whenever a local deviation is required, it is stated explicitly in the relevant section.

**0.4 Core Proposal**

At its core, Absolute Relativity (AR) is a qualia-first, relational theory in which everything—spacetime, fields, particles, and gravity—emerges from a discrete algebra of present-moment updates and their nested context structure.

The primitive objects are **ticks** and **Present-Moment Spheres** (PMS): each tick carries a PMS that splits into an Inner Network (IN, committed record) and an Outer Network (ON, admissible potential). A small alphabet of primitive operators—Renew, Sink, Trade, Sync, and a Framing composite—acts on these PMS-carriers. A ledger attached to each carrier tracks how much capacity has become record and how much remains potential, and the non-commutation of these operators, together with ledger monotonicity, generates an intrinsic arrow of time. From the statistics of flip counts we construct three interval components that satisfy a Lorentz-like invariant relation, so relativity-style cones and time dilation appear as consequences of the tick algebra rather than as background structure.

The IN side of the PMS is modeled as a fractal attractor with effective dimension and a pivot function normalized at a hinge value . Dual exponents for inward “record thickening” and outward “potential spread” define an effective present-moment dimension close to two, singling out the PMS boundary as a hinge between volume-like and filamentary regimes. A two-dimensional present plane with a complex structure carries present amplitudes, and a structural Born-style rule ties squared amplitudes to a reference measure on IN outcome basins. A context-gate rule based on distinguishes full-strength interactions within a Collective Sphere (CS) from attenuated interactions across context boundaries.

Above this, the theory introduces a **discrete context ladder**, indexed by integers , with a dimension curve sampled from a radial profile and pivot weights . Collapse and expansion operators connect neighboring bands via boundary Hilbert spaces, and a reproduction kernel on each band defines a memory dimension . A master action on the ladder, weighted by the pivot factors, admits a continuum embedding in context-time; its Euler–Lagrange and Hamiltonian forms give field-like equations for band fields, and a canonical quantization scheme yields path-sums over ladder histories. CS-sampling maps these ladder amplitudes back to the hinge, recovering Born-style probabilities and standard quantum-mechanical behavior as structural outputs of the algebra.

On this scaffold we construct **gauge, matter, and gravitational sectors**. U(1) and non-Abelian gauge structures arise from link variables on boundary graphs and pivot-weighted Yang–Mills-type actions; matter spectra appear as eigenstructures of ladder/gauge operators with band-localized profiles. A gravitational sector emerges from the hinge and pivot profile: area-law horizons and inverse-square fields are derived as relational properties of the hinge band, and the full ladder induces scale-dependent deviations from pure behavior. Collapse kernels on the PMS boundary (S²) and on null-cone shells, together with mirror-symmetry functionals on , single out the zero-context band as a unique pivot, while a retarded composite operator on null shells leads to a unified inverse-square law and a compact 4D scalar field equation whose limiting forms resemble Maxwell and Poisson equations.

Finally, we show that these structures admit an abstract **present-act engine** realization. In this engine (spelled out structurally in Section 12), local neighbor transitions are decided by a finite feature alphabet, boolean/ordinal gates, and a ratio-lexicographic acceptance rule; typed budgets are read out from committed acts and obey the same invariant interval derived at the tick level. Gravity appears as a rotation-invariant feasibility geometry inside this engine, implemented via a radial gate family and nested-sphere constructions that reproduce the hinge-based inverse-square behavior already encoded in the ladder and pivot structure.

In summary, the proposal is that a single discrete, relational machinery—ticks, PMS/IN/ON, the flip algebra, the fractal hinge at , the context ladder and pivot function, and their abstract engine realization—suffices to recover the familiar kinematics and field behavior of relativistic quantum theory and gravity, without postulating an independent spacetime manifold or continuous fields as primitives.

**0.5 Document Roadmap**

This volume is organized so that each layer of the theory builds directly on the previous one, moving from primitive objects and operators up to the unified 4D field structure. What follows is a brief guide to the main parts and how they fit together.

**0.5.1 Part I – Primitive Ontology & Present-Moment Architecture (Section 1)**

Part I introduces the **basic objects** of the theory:

* Ticks as atomic updates.
* Present-Moment Spheres (PMS) with Inner (IN) and Outer (ON) networks.
* Collective Spheres (CS) as synchronized collections of PMSs.
* An informal view of the context ladder indexed by integers (n).

The goal of Part I is to define *what exists* in the model before any algebra or geometry is introduced.

**0.5.2 Part II – Formal Preliminaries & Tick-State Carriers (Section 2)**

Part II sets up the **minimal formal language**:

* The tick set, carrier objects (\mathcal{C}\_k), and the abstract state space (\mathcal{H}).
* Domains and ranges for operators acting on carriers.
* Flip words (operator compositions) and flip-count vectors.

This part supplies just enough structure to state precise algebraic laws in the next section.

**0.5.3 Part III – Tick-Operator Algebra, Arrow, and Interval (Section 3)**

Part III develops the **core operator algebra**:

* Definitions of the primitive operators (F, S, T, C, CT).
* Ledger structure and monotonicity properties that encode an arrow of time.
* Construction of the invariant interval from flip counts and its Lorentz-like form.
* The role of CS as frames and the associated symmetry group.

After Part III, an expert reader can see how a spacetime-like kinematics is derived purely from operator structure.

**0.5.4 Part IV – Fractal Inner Geometry & Pivot Gate (Section 4)**

Part IV introduces the **fractal structure of IN** and the **pivot function**:

* IN as an iterated-function-system (IFS) attractor with effective dimension (D).
* Pivot function (g(D)) and its constraints (including (g(2)=1)).
* Dual time exponents for record and potential and the effective present dimension near 2.
* A 2D present plane with complex structure (J).
* A context gate rule that uses (g(D)) to weight couplings.

This part connects the algebra of flips to a geometric picture of how “thickness” and coupling strength depend on dimension.

**0.5.5 Part V – Unified Fractal–Sink Paradigm & Radial Profile (D(r)) (Section 5)**

Part V unifies the **sink/renew dynamics** with the **radial fractal profile**:

* Radial or context variable (r) with a hinge at (r=0).
* Dimension profile (D(r) = 2 + \delta(r)) and its qualitative behavior.
* The boundary projector (\mathcal{B}) and its role at the hinge.
* The double-flip-plus-projection theorem at the pivot.
* Cross-scale constraints on the pivot function (g(D)) and nested CS ladders.

This part refines the notion of a “hinge band” where the theory’s key symmetries and simplifications occur.

**0.5.6 Part VI – Discrete Context Ladder, Reproduction Kernel & (D(n)) (Section 6)**

Part VI formalizes the **discrete context ladder**:

* Integer-indexed context bands (n) and their boundary graphs.
* Collapse and expansion operations between bands.
* The reproduction kernel and memory dimension (D\_{\mathrm{mem}}(n)).
* Logistic-style dimension curves (D(n)) with a hinge at (n=0).
* The relation between (D(n)), (D(r)), and the pivot weights (g(D(n))).

Here the ladder becomes an explicit mathematical object that will support actions and quantization.

**0.5.7 Part VII – Master Action, Quantization & Renormalization (Section 7)**

Part VII defines the **dynamical layer** on the ladder:

* A discrete master action over context bands, weighted by (g(D(n))).
* Continuum embedding in context-time and the associated Lagrangian and Hamiltonian forms.
* A quantization scheme expressed as path-sums over flip histories.
* CS sampling and recovery of Born-like probability weights.
* A renormalization-group interpretation in terms of context flips and flows of (D(n)), (g(D)).

By the end of this part, the theory has a complete kinematic and dynamic structure at the level of the context ladder.

**0.5.8 Part VIII – Context–Unit Dictionary & Hinge Scales (Section 8)**

Part VIII provides a **symbolic mapping** between context indices and physical units:

* Structural mapping from context bands to coarse-grained temporal scales.
* Definition of the spatial hinge length (\ell\_{\mathrm{UGM}}) and temporal hinge (T^\*) as geometric-mean–type constructs.
* The relation of these hinges to the invariant interval and the constant (c).
* A minimal dictionary relating ladder bands to typical domains (particle, mesoscopic, macroscopic, astrophysical) without fixing numerical values.

This part remains theory-only but clarifies how the abstract ladder would correspond to familiar “sizes.”

**0.5.9 Part IX – Gauge Structure & Matter Spectrum (Section 9)**

Part IX builds **gauge and matter sectors** on top of the ladder:

* U(1) link variables, Wilson loops, and pivot-weighted actions.
* Extension to non-Abelian gauge groups (SU(2), SU(3)) and Yang–Mills-type actions.
* Context eigenstates and projectors that encode matter spectra at the structural level.
* Gauge bosons and mixing matrices as features of context connections.

No numeric parameters are fixed; this part specifies how such sectors would arise formally from the ladder.

**0.5.10 Part X – Gravitational Sector & Nested-Time Matter (Section 10)**

Part X develops the **gravitational sector** and formalizes **nested-time matter**:

* Effective gravitational field equations from the hinge and pivot profile.
* Structural inverse-square behavior and its scale-dependent deviations.
* Horizon formation, entropy, and area-law scaling in ladder language.
* Nested-time matter across different bands and the formal link between inner (e.g. neural) and outer (gravitational) ladders.

This part ties the purely algebraic and geometric structures back into something recognizable as “gravity” without appealing to data.

**0.5.11 Part XI – Collapse Kernels, Light-Cone Geometry & Unified Inverse-Square Law (Section 11)**

Part XI focuses on **collapse kernels** and **light-cone geometry**:

* Collapse kernels on the PMS boundary (S^2) and their behavior at (D=2).
* Mirror-symmetry functionals that single out the pivot band.
* Fractal interpretation of null-cone shells and a 4D hyperspherical collapse kernel.
* A retarded composite operator that reduces to identity at the pivot.
* Structural derivation of unified inverse-square behavior from the hinge.

This provides the bridge from ladder geometry to light-cone–like structures.

**0.5.12 Part XII – Present-Act Engine & Feasibility Geometry (Section 12)**

**Part XII develops an abstract present-act realization of the AR machinery and recasts gravity as a feasibility geometry.** Section 12 takes the objects and invariants defined earlier—ticks, PMS/IN/ON, flip algebra, hinge scales , and the ladder/pivot structure—and packages them into a tick-local engine contract. The engine operates on discrete neighbor transitions between candidate world/qualia states, using a finite feature alphabet, boolean/ordinal gates, and a ratio-lexicographic acceptance rule. Typed budgets are read out after each commit and obey the same invariant interval introduced in Part III.

On top of this engine, Part XII defines **gravity as a rotation-invariant feasibility geometry**: a radial gate family (ParentGate) acts on concentric shells, and a unit-free gravity amplitude is built from shell-wise survival fractions. At the hinge, this construction reproduces inverse-square behavior and area-law signatures consistent with the hinge-based gravitational sector of Part X and the unified inverse-square structure of Part XI. Nested-sphere IFS constructions on yield an onion of concave/convex shells and horizon templates, making explicit how the ladder/pivot picture can be realized as a present-act engine without introducing any implementation-level details, numerical schemes, or empirical calibration.

**0.5.13 Part XIII – Indices & Cross-Reference Maps (Section 13)**

Finally, Part XIII collects **indices and cross-reference tools**:

* A definitions index (DEF).
* An axioms index (AX).
* A propositions/theorems index (PROP/THM).
* A notation index.

These indices make it possible to treat the volume as a reference manual for the V1 AR formalism, with every core object and statement locateable by ID and section number.

**1. Primitive Ontology & Present-Moment Architecture**

**1.1 Qualia-First Tick Premise**

The V1 formalism of Absolute Relativity starts from a minimal assumption about what is taken as **primitive**. Instead of postulating a pre-existing spacetime in which “events” occur, the theory takes as basic a discrete family of **present-moment updates**, which we call **ticks**.

The word “qualia” is used to emphasize that these ticks are meant to represent units of *lived* present-moment content rather than abstract points in a geometric background. In this volume, however, we do **not** attempt to analyse or defend any philosophical position about consciousness; “qualia-first” here serves only as a label for the following structural choice:

* We treat **present-moment experiences** as the primary objects.
* We treat spacetime, trajectories, and fields as **derived** structures built from relations among these experiences.

Formally, we proceed as follows.

1. **Tick set.**  
   We assume a countable set of tick labels  
   [  
   T = { k \mid k \in \mathbb{Z} },  
   ]  
   where each (k) indexes one atomic present-moment update. No geometric meaning is assigned to (k) at this stage; it is simply a discrete ordering parameter.
2. **Underlying sequence.**  
   The theory is concerned with sequences of ticks  
   [  
   \dots, k-1, k, k+1, \dots  
   ]  
   together with the structures attached to each tick. The intuitive reading is that “something” is updated from one present to the next, but the formalism does not presuppose any particular substrate (neural, physical, or otherwise).
3. **Present-moment content as primitive.**  
   For each tick (k), there exists a **Present-Moment Sphere** (PMS(\_k)), which captures the full “content” of that tick in the sense relevant to the theory. PMS(\_k) will later be decomposed into an Inner Network (record) and an Outer Network (potential), but at this stage it is simply taken as a primitive container associated with (k).
4. **No background spacetime.**  
   The theory does **not** assume a pre-given spacetime manifold on which ticks live. Instead:
   * Ordering relationships (earlier/later) arise from the allowed compositions of the primitive operators acting on ticks.
   * Metric relationships (durations, distances) arise from **invariants** of these operator compositions (flip counts), not from a pre-imposed geometry.
5. **Reality as relational network of ticks.**  
   The “world” within this formalism is the network formed by:
   * The family of ticks (k).
   * The structures attached to each tick (PMS(\_k) and, later, IN(\_k), ON\_k)).
   * The relations between ticks generated by the primitive operators (Renew, Sink, Trade, Sync, Framing).
6. **Role of the qualia-first language.**  
   Informally, one can say that each tick is “what it is like” to be in a certain present state, but in this volume:
   * We do not analyse this notion further.
   * We do not impose any particular phenomenological structure beyond what is needed to define PMS, IN, and ON.
   * We allow all further constructions (intervals, fields, ladders) to be developed from the algebra and geometry defined on these primitive objects.

In summary, Section 1.1 fixes the basic stance of the theory: **discrete present-moments (ticks) and their associated spheres of content are taken as fundamental**, and everything usually described in terms of spacetime and fields is to be reconstructed from the relations among these ticks. Subsequent subsections will refine this picture by specifying how PMSs are structured (IN/ON), how multiple PMSs synchronize into Collective Spheres (CS), and how the informal notion of a “context ladder” is introduced.

**1.2 Present-Moment Sphere (PMS)**

In the V1 formalism, each tick (k) carries a **Present-Moment Sphere** (\mathrm{PMS}\_k). This is the basic “container” for all structure that is relevant at tick (k). The PMS is not embedded in a prior spacetime; instead, it is the object from which spatial and temporal notions will eventually be derived.

**1.2.1 Informal role**

At an informal level:

* (\mathrm{PMS}\_k) represents “everything that is jointly present” at tick (k), in the sense of the model.
* It splits into:
  + an **Inner Network** (\mathrm{IN}\_k), encoding committed record;
  + an **Outer Network** (\mathrm{ON}\_k), encoding admissible potential.
* Later, the PMS boundary will be treated as a distinguished 2D surface (at the hinge dimension (D=2)) where collapse kernels act and where gauge and gravitational structures are naturally defined. Here we only fix the abstract structure.

**1.2.2 Formal definition**

We now specify the PMS at tick (k) as a structured pair attached to that tick.

**Definition 1.2.1 (Present-Moment Sphere).**  
For each tick (k \in \mathbb{Z}), the **Present-Moment Sphere** is a pair  
[  
\mathrm{PMS}\_k = (\mathrm{IN}\_k,\mathrm{ON}\_k),  
]  
where:

1. (\mathrm{IN}\_k) (the **Inner Network**) is a structured object representing the committed record at tick (k). Formally, one may regard (\mathrm{IN}\_k) as:
   * a labeled graph,
   * or a subset of a state space (\mathcal{H}),
   * or an attractor of an iterated-function system,  
     depending on context. In what follows, we treat (\mathrm{IN}\_k) abstractly as “the record side” without committing to a specific representation until later sections.
2. (\mathrm{ON}\_k) (the **Outer Network**) is a structured object representing the admissible potential at tick (k): the collection of branches or continuations that are compatible with the current record and the operator algebra.

We impose the following minimal constraints:

* **(Coverage)** All relational content relevant to tick (k) is encoded in (\mathrm{IN}\_k) and (\mathrm{ON}\_k); there is no third region.
* **(Disjointness of roles)** While (\mathrm{IN}\_k) and (\mathrm{ON}\_k) may be defined on the same underlying carrier sets (e.g. the same nodes or locations), they are distinguished by role: elements counted as “record” at tick (k) are not counted as “potential” at the same tick.
* **(Tick attachment)** Each PMS is attached to a single tick label (k); any change in (\mathrm{IN}) or (\mathrm{ON}) corresponds to moving to another tick (k').

With this definition, the PMS does not yet carry any explicit geometric structure (metric, dimension, etc.). Those properties are introduced later when we model (\mathrm{IN}\_k) as a fractal object and consider its effective dimension (D).

**1.2.3 PMS and carriers**

The PMS appears concretely in the **carrier** attached to tick (k).

**Definition 1.2.2 (Carrier at tick (k)).**  
A **carrier** at tick (k) is a tuple  
[  
\mathcal{C}\_k = (k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k),  
]  
where:

* (k \in \mathbb{Z}) is the tick label;
* (h\_k \in \mathcal{H}) is an abstract state (e.g. a vector or configuration) representing the current “compressed” state of the system relevant to the algebra of operators;
* ((\mathrm{IN}\_k,\mathrm{ON}\_k)) is the Present-Moment Sphere (\mathrm{PMS}\_k).

The primitive operators introduced later (Renew, Sink, Trade, Sync, Framing) act on carriers (\mathcal{C}\_k), updating both the tick index and the PMS components. Schematically, an operator (O) maps  
[  
O : \mathcal{C}*k \mapsto \mathcal{C}*{k'},  
]  
with the PMS part transformed as  
[  
(\mathrm{IN}*k,\mathrm{ON}k) \mapsto (\mathrm{IN}{k'},\mathrm{ON}*{k'}).  
]

**1.2.4 PMS as boundary between record and potential**

Although we treat (\mathrm{IN}\_k) and (\mathrm{ON}\_k) abstractly at this stage, it is useful to fix the **boundary interpretation**:

* (\mathrm{IN}\_k) is the part of the structure that is already fixed (record).
* (\mathrm{ON}\_k) is the part that remains open (potential).
* The PMS boundary at tick (k) is the conceptual interface across which potential structure becomes record as ticks advance.

Later, when we introduce fractal geometry and collapse kernels, this boundary will be modeled as a 2-dimensional surface (at hinge dimension (D=2)) with specific properties (e.g. area-law behavior, constant-collapse at the pivot). For now, we only use the language of “sphere” and “boundary” as a reminder that:

* The PMS is meant to be a **bounded** present context, with an interior (record) and an exterior (potential).
* The primitive operators will be defined in such a way that they move content from (\mathrm{ON}\_k) to (\mathrm{IN}\_k) (and possibly back out under certain constructions), implementing the transition from potential to record at each tick.

**1.2.5 Summary of the PMS role**

Section 1.2 has fixed the PMS as:

* a **tick-attached primitive** object (\mathrm{PMS}\_k = (\mathrm{IN}\_k,\mathrm{ON}\_k));
* the **boundary** between record and potential at tick (k);
* the **place** where later geometrical structures (dimension, pivot, collapse kernels) are defined.

In the next subsection, we will refine the Inner/Outer distinction itself, and then introduce Collective Spheres (CS) as synchronized collections of PMSs that will serve as frames in the later kinematic and field-theoretic constructions.

**1.3 Inner and Outer Networks (IN/ON)**

The decomposition of each Present-Moment Sphere into an **Inner Network** and an **Outer Network** is the core structural split of the V1 formalism. Intuitively, the inner side represents what is already **fixed** (record), while the outer side represents what remains **open** (potential). In this section we make that distinction precise in the minimal way needed for later algebra and geometry.

**1.3.1 Informal role of IN and ON**

At a high level:

* (\mathrm{IN}\_k) is the portion of the present-moment structure at tick (k) that:
  + has already been “committed”,
  + is stable under admissible evolutions,
  + and is treated as **record** by the theory.
* (\mathrm{ON}\_k) is the portion of the present-moment structure at tick (k) that:
  + remains open to multiple continuations,
  + has not yet been resolved into record,
  + and is treated as **potential** by the theory.

Future sections will specify how the primitive operators move content from (\mathrm{ON}) into (\mathrm{IN}) (and exchange boundary structure), but the basic picture is:

* **Sink-like operations** grow (\mathrm{IN}\_k) at the expense of (\mathrm{ON}\_k).
* **Renew-like operations** enlarge or refresh (\mathrm{ON}\_k), altering what is admissible as potential.
* **Trade-like operations** redistribute structure between (\mathrm{IN}\_k) and (\mathrm{ON}\_k) while preserving certain budgets.

**1.3.2 Formal definition of IN and ON**

We now define (\mathrm{IN}\_k) and (\mathrm{ON}\_k) as abstract structured objects attached to each tick (k).

**Definition 1.3.1 (Inner and Outer Networks).**  
For each tick (k\in\mathbb{Z}), let  
[  
\mathrm{PMS}\_k = (\mathrm{IN}\_k,\mathrm{ON}\_k)  
]  
be the Present-Moment Sphere at tick (k). Then:

1. The **Inner Network** (\mathrm{IN}\_k) is a structured object satisfying:
   * It is closed under the “record update” relations defined by the primitive operators (later sections will specify these relations in detail).
   * It can be regarded as an element of some abstract configuration space (\mathcal{I}) (e.g. graphs, states, or subsets of (\mathcal{H})), but this volume does not fix a particular representation.
2. The **Outer Network** (\mathrm{ON}\_k) is a structured object satisfying:
   * It encodes admissible continuations compatible with (\mathrm{IN}\_k) and with the current tick-state (h\_k).
   * It can be regarded as an element of some abstract configuration space (\mathcal{O}), again without fixing a specific representation.

We impose the following high-level constraints:

* **(Role separation)**  
  At any given tick (k), the roles of record and potential are disjoint:  
  [  
  x \in \mathrm{IN}\_k ;\Rightarrow; x \not\in \mathrm{ON}\_k  
  ]  
  for any element (x) in the underlying carrier set (where “element” is interpreted appropriately for the chosen representation).
* **(Completeness with respect to relevance)**  
  All structure that is relevant to tick (k) from the standpoint of the theory is included in either (\mathrm{IN}\_k) or (\mathrm{ON}\_k). There is no third region.
* **(Tick-indexed evolution)**  
  Any change in (\mathrm{IN}) or (\mathrm{ON}) corresponds to moving from tick (k) to some (k'\neq k); that is, the PMS is updated via the action of operators, not continuously within a single tick.

**1.3.3 Record monotonicity and potential restriction**

Although the detailed ledger and operator laws appear later, it is useful to state here two structural properties that IN and ON are required to satisfy along admissible tick sequences.

Let  
[  
\mathcal{C}\_k = (k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k)  
]  
be the carrier at tick (k), and consider an admissible operator word (\Pi) that takes (\mathcal{C}*k) to some (\mathcal{C}*{k'}).

We impose:

* **(Record monotonicity)**  
  The inner network is monotone in the sense that along any admissible evolution  
  [  
  \mathrm{IN}*k \subseteq \mathrm{IN}*{k'},  
  ]  
  where inclusion is in whatever structural sense is appropriate (e.g. as a subgraph, subset, or coarse-graining). Intuitively, record can grow or reorganize, but it cannot “forget” in ways that violate the underlying ledger axioms.
* **(Potential restriction)**  
  The outer network evolves under constraints that depend on the primitive operators. Schematically:
  + Sink-like operations reduce the admissible potential in (\mathrm{ON}\_k).
  + Renew-like operations can increase or reset portions of (\mathrm{ON}\_k).
  + Trade-like operations move elements across the (\mathrm{IN}/\mathrm{ON}) boundary while respecting capacity constraints.

In later sections, these informal statements are made precise via:

* budget functions (I(\mathcal{C}\_k)) and (E(\mathcal{C}\_k)) associated with (\mathrm{IN}\_k) and (\mathrm{ON}\_k), respectively;
* algebraic laws for how (F, S, T, C, CT) update (\mathrm{IN}\_k), (\mathrm{ON}\_k), and the ledger.

**1.3.4 Compatibility with the abstract state (h\_k)**

Recall that each carrier (\mathcal{C}\_k) includes an abstract state (h\_k\in\mathcal{H}). We require that:

* (\mathrm{IN}\_k) and (\mathrm{ON}\_k) are **compatible** with (h\_k) in the sense that:
  + (h\_k) encodes a compressed or coarse-grained description of the inner and outer networks; and
  + all admissible operator actions on (h\_k) (as elements of (\mathcal{H})) are consistent with the allowed transformations of (\mathrm{IN}\_k) and (\mathrm{ON}\_k).

Formally, there is a (possibly many-to-one) map  
[  
\Phi\_k : (\mathrm{IN}*k,\mathrm{ON}k) \mapsto h\_k  
]  
such that if an operator (O) is admissible on (\mathcal{C}k), then the transformed carrier (\mathcal{C}{k'} = O(\mathcal{C}k)) satisfies  
[  
\Phi{k'}(\mathrm{IN}{k'},\mathrm{ON}*{k'}) = \hat{O}(h\_k),  
]  
where (\hat{O}) is the corresponding operator on (\mathcal{H}). The details of (\hat{O}) and (\Phi\_k) are not fixed here; this condition simply expresses that the PMS-level description and the abstract state level are coherent.

**1.3.5 Relationship to future geometric structure**

Later in the volume, additional structure will be imposed on (\mathrm{IN}\_k) and its boundary:

* (\mathrm{IN}\_k) will be modeled as a fractal attractor of an iterated-function system, with an effective dimension (D).
* The PMS boundary (where IN meets ON) will be treated as a 2-dimensional surface when (D=2), and collapse kernels will act on this boundary.
* The pivot function (g(D)) will control how strongly different contexts (with different IN dimensions) couple.

For now, we deliberately keep (\mathrm{IN}\_k) and (\mathrm{ON}\_k) as abstract as possible. They are:

* **Typed containers** for record and potential at each tick,
* Subject to monotonicity and compatibility constraints,
* And designed so that later geometrical and algebraic constructions can be layered on without changing the core definitions.

The next subsection will extend this picture from individual PMSs to **Collective Spheres (CS)**—synchronized clusters of PMSs that act as frames and provide the scaffolding for relativistic and quantum structures later in the theory.

**1.4 Collective Spheres (CS)**

Individual PMSs capture what is present for a single tick-indexed locus. The theory also needs a way to represent **shared present structure** across many PMSs at once. This role is played by **Collective Spheres (CS)**: synchronized clusters of PMSs that act as frames in later kinematic and field-theoretic constructions.

**1.4.1 Informal role**

Informally, a Collective Sphere:

* Groups together many PMSs whose **present-moment structure is mutually compatible**.
* Acts as a **reference frame**: within a CS, ticks can be compared and coordinated, and certain quantities are naturally taken as “the same” across members.
* Provides the level at which **frame symmetries** (e.g. Lorentz-like transformations) and **collective measurements** (e.g. objectification of outcomes) are expressed.

Later, when we discuss measurement and quantum structure, strong framing operations will lock a set of PMSs into a CS so that a single outcome is effectively shared within that group.

**1.4.2 Formal definition**

We now define Collective Spheres in the minimal form needed for the rest of the volume.

**Definition 1.4.1 (Collective Sphere).**  
Let ({ \mathrm{PMS}*{k,\alpha} }*{\alpha \in A}) be a family of Present-Moment Spheres indexed by some set (A) (for example, different loci, subsystems, or agents), and let (k) be a shared tick label. A **Collective Sphere at tick (k)** is specified by:

* A non-empty index set (A) of PMS labels.
* A PMS for each (\alpha \in A):  
  [  
  \mathrm{PMS}*{k,\alpha} = (\mathrm{IN}*{k,\alpha}, \mathrm{ON}\_{k,\alpha}).  
  ]
* A **synchronization condition** that all PMSs in the set are mutually consistent with a common effective boundary configuration.

We write such a CS as  
[  
\mathrm{CS}\_k(A)  
]  
or simply (\mathrm{CS}) when context makes (k) and (A) clear.

The synchronization condition is left abstract here, but it must at least satisfy:

* There exists a **shared effective boundary description** (B\_k) such that for every (\alpha\in A), the PMS boundary of (\mathrm{PMS}\_{k,\alpha}) is compatible with (B\_k) (e.g. by projecting onto a common coarse-grained representation).
* The operator (C) (Sync) introduced later acts to enforce or maintain this compatibility when it is applied to the relevant carriers.

Thus, a CS is not just a set of PMSs at the same tick index; it is a **structured collection** that admits a single consistent description of “what is present” at the boundary level.

**1.4.3 CS carriers and the Sync operator**

To connect CS to the carrier formalism, we introduce a **CS carrier** as an aggregate of individual carriers.

**Definition 1.4.2 (CS carrier).**  
Given carriers  
[  
\mathcal{C}*{k,\alpha} = (k, h*{k,\alpha}, \mathrm{IN}*{k,\alpha}, \mathrm{ON}*{k,\alpha})  
]  
indexed by (\alpha \in A), a **CS carrier** at tick (k) is an object  
[  
\mathcal{C}*k^{\mathrm{CS}}(A) = \bigl(k, { h*{k,\alpha} }*{\alpha \in A}, { \mathrm{IN}*{k,\alpha} }*{\alpha \in A}, { \mathrm{ON}*{k,\alpha} }\_{\alpha \in A} \bigr)  
]  
together with the synchronization condition required for (\mathrm{CS}\_k(A)).

The **Sync operator** (C) (defined in detail later) can be viewed in two equivalent ways:

* As an operator on individual carriers that tends to **align** their PMS boundaries so they can be collected into a CS.
* As an operator acting on a CS carrier to **maintain or refine** synchronization, e.g.  
  [  
  C : \mathcal{C}*k^{\mathrm{CS}}(A) \mapsto \mathcal{C}*{k'}^{\mathrm{CS}}(A'),  
  ]  
  potentially changing the tick label (from (k) to (k')) and the index set (from (A) to (A')) while preserving or tightening shared boundary structure.

The detailed algebra of (C) is given in later sections; here we only set up the CS as the object on which (C) acts collectively.

**1.4.4 CS as frames and symmetry groups**

Collective Spheres are the natural habitats for **frame symmetries**:

* Within a CS, we can define quantities such as (\Delta t, \Delta \tau, |\Delta x|) for relationships between events associated with different carriers.
* Transformations that act on all PMSs in a CS while preserving the invariant interval form a symmetry group, denoted (\mathcal{G}\_{\mathrm{CS}}).

At the formal level:

* (\mathcal{G}\_{\mathrm{CS}}) is the group of transformations on CS carriers that:
  + preserve the tick-ordering structure (up to relabeling),
  + preserve the invariant interval,
  + and respect the synchronization condition of the CS.

We do not specify a particular matrix representation of (\mathcal{G}\_{\mathrm{CS}}) in this section. The important point is that **frames** in AR are tied to **CS structures**: a “frame” is nothing more than a CS endowed with its symmetry group.

**1.4.5 Strong and weak framing (preview)**

Later, when we discuss the Framing operator (CT) and measurement-like phenomena, it will be important to distinguish:

* **Strong framing**: operations that force a set of PMSs into such tight synchronization that they share an effectively unique outcome at the CS level (e.g. objectification of a measurement result across an apparatus and an observer).
* **Weak framing**: operations that partially correlate PMSs without fully eliminating alternatives in (\mathrm{ON}).

In both cases, the CS is the **carrier** of these relations:

* Strong framing maps a broad class of PMS collections into a CS with sharply constrained IN-structures.
* Weak framing modifies (\mathrm{ON})-structures across a CS in a way that biases but does not uniquely fix future record.

This section does not formalize strong/weak framing; it simply fixes the idea that CSs are the **shared present contexts** on which such framing acts.

**1.4.6 Summary of CS role**

Section 1.4 has introduced Collective Spheres as:

* Synchronized collections of PMSs with mutually consistent boundary descriptions.
* The natural **frame objects** in the AR formalism, supporting symmetry groups and shared invariants.
* The targets of the Sync operator (C) and, later, the Framing composite (CT).

In the next subsection (1.5), we move from the notion of multiple PMSs at one “scale” to an **informal context ladder**, where PMSs and CSs are organized into bands indexed by integers (n) that will later be tied to effective dimensions and pivot weights.

**1.5 Context Ladder (Informal)**

So far we have described PMSs and CSs at a single, undifferentiated “level.” The full AR framework, however, requires an explicit way to talk about **nested** and **coarse-grained** descriptions of reality. This is the role of the **context ladder**.

In this subsection we introduce the idea informally. The full formal machinery (dimension curves, reproduction kernels, actions) appears later in Parts V–VII.

**1.5.1 Context bands and the index (n)**

We assume that present-moment structure can be described at multiple **context bands**:

* A context band is labeled by an integer  
  [  
  n \in \mathbb{Z}, \quad \dots, -2, -1, 0, +1, +2, \dots  
  ]
* Each band (n) corresponds to a way of **organizing** PMSs and CSs:
  + Negative indices ((n<0)) represent **inner** or more fine-grained contexts.
  + Positive indices ((n>0)) represent **outer** or more coarse-grained contexts.
  + The band (n=0) will later be singled out as a **hinge** or **pivot** band.

At this stage, (n) is just an abstract label; we do not yet attach any geometric scale or fractal dimension to it. The key point is that we allow many descriptions of “the same situation” at different context bands.

**1.5.2 PMSs and CSs within a context band**

For each context band (n), we can consider:

* PMSs that are described **at that band**, e.g. PMSs whose IN/ON structure is represented at a certain level of coarse-graining.
* CSs that are formed from PMSs at that band.

Informally:

* **Within a fixed (n)**, we can talk about a family of PMSs and CSs as we already have in Sections 1.1–1.4.
* The context index tells us **how much structural detail** is being treated as relevant at that level.

Later, when we introduce the dimension curve (D(n)), the band (n) will also carry information about the effective fractal dimension of the IN structure at that level.

**1.5.3 Inner vs outer contexts**

The sign of (n) has a qualitative meaning:

* (n<0) (inner bands):
  + Emphasize **sub-structures** that are experienced as “within” the present environment.
  + For example, in informal terms: micro-structure, internal degrees of freedom, finer-scale organization.
* (n=0) (hinge band):
  + Represents the **present-moment environment itself**, in the sense that will be made precise later.
  + This band often corresponds to the “centered” viewpoint in which PMS/CS are directly described.
* (n>0) (outer bands):
  + Emphasize **super-structures** that are experienced as “around” or “containing” the present environment.
  + For example, larger-scale organization, collective or ambient structure.

No empirical scale (e.g. in meters or seconds) is attached here; we only assert that there is a structured *ordering* of contexts from inner to outer, with a special role for (n=0).

**1.5.4 Embedding and nesting relationships**

Conceptually, each PMS participates in multiple context bands through different descriptions:

* A PMS at band (n=0) has:
  + **Inner descriptions** at (n=-1,-2,\dots) that refine its internal structure.
  + **Outer descriptions** at (n=+1,+2,\dots) that place it within larger CSs and environments.

We do not yet formalize the maps between bands, but we assume:

* There is a **nesting relationship** such that descriptions at (n<0) can be regarded as “inside” those at (n=0), and descriptions at (n>0) can be regarded as “around” or “containing” those at (n=0).
* These relationships will later be encoded via:
  + A reproduction kernel linking IN structures across bands.
  + Dimension curves (D(n)) and pivot weights (g(D(n))).
  + Collapse/expansion operators that move information between bands.

At this informal stage, the context ladder is simply a way of saying: *the present-moment world can be viewed at many nested levels, and we index these levels by (n).*

**1.5.5 Hinge intuition (without geometry)**

Even before introducing explicit dimensions, it is useful to record the **hinge intuition**:

* The band (n=0) is special in that:
  + It corresponds to the **present environment** from which we measure inward and outward.
  + It will later be associated with an effective fractal dimension (D(0)=2) and a pivot weight (g(D(0))=1).
  + Collapse kernels and certain symmetry properties simplify at this band.

For now, we only mark (n=0) as a distinguished index. Its geometric and dynamical significance will be developed in later sections (particularly 5–7 and 11–12).

**1.5.6 Summary**

Section 1.5 has introduced the context ladder in purely informal terms:

* A discrete set of context bands indexed by integers (n).
* A qualitative distinction between inner ((n<0)), hinge ((n=0)), and outer ((n>0)) descriptions.
* The idea that PMSs and CSs can be represented at different bands, corresponding to nested or coarse-grained views.

In Part II, we will shift from this conceptual picture to the minimal formal machinery (carriers, operator domains, flip words) needed to state the algebraic laws that govern how present-moment structures evolve tick by tick.

**2. Formal Preliminaries & Tick-State Carriers**

**2.1 Basic Sets & Maps**

This section introduces the minimal formal ingredients needed for the later algebra: the tick set, ordering and separation maps on ticks, and a few basic constructions that we will reuse throughout the volume. The goal is simply to make precise *what kind of objects* our operators act on and how tick indices are organized.

**2.1.1 Tick set and ordering**

We begin by fixing the set of tick labels.

**Definition 2.1.1 (Tick set).**  
The **tick set** is  
[  
T := { k \mid k \in \mathbb{Z} },  
]  
whose elements label atomic present-moment updates.

We assume:

* A **total order** (\leq) on (T) given by the usual order on (\mathbb{Z}).
* A **successor map** (s : T \to T) defined by  
  [  
  s(k) = k+1.  
  ]
* A **predecessor map** (p : T \to T) defined by  
  [  
  p(k) = k-1.  
  ]

These maps are used only as convenient shorthand; the theory does not assume any underlying continuous parameter behind the discrete ticks.

**2.1.2 Separation and composition of ticks**

We now define a simple separation map and a composition operation on tick differences.

**Definition 2.1.2 (Separation map).**  
The **separation** between ticks (k\_1, k\_2 \in T) is the integer  
[  
\Delta(k\_2,k\_1) := k\_2 - k\_1.  
]

This satisfies:

* (\Delta(k,k) = 0),
* (\Delta(k\_3,k\_1) = \Delta(k\_3,k\_2) + \Delta(k\_2,k\_1)),
* (\Delta(k\_2,k\_1) = -\Delta(k\_1,k\_2)).

We will often write (\Delta k := \Delta(k\_2,k\_1)) when the endpoints are clear from context.

**Definition 2.1.3 (Tick-difference monoid).**  
Let  
[  
\mathbb{T}\_\Delta := (\mathbb{Z}, \oplus)  
]  
denote the additive monoid of tick differences, where the operation (\oplus) is just integer addition:  
[  
a \oplus b := a + b.  
]

Tick separations are thus elements of (\mathbb{T}\_\Delta); their composition obeys the usual associative law and has identity element (0).

**2.1.3 Present-moment index sets**

At various points, we will work with finite or countable collections of PMSs and carriers. For this, we fix some basic notation.

* If (A) is any set of labels (e.g. indexing subsystems, observers, or loci), then  
  [  
  { \mathrm{PMS}*{k,\alpha} }*{\alpha \in A}  
  ]  
  denotes a family of PMSs attached to the same tick (k), indexed by (\alpha \in A).
* Similarly,  
  [  
  { \mathcal{C}*{k,\alpha} }*{\alpha \in A}  
  ]  
  denotes the corresponding family of carriers at tick (k).

No additional structure is imposed on (A) beyond what is explicitly specified when needed (e.g. finite size, partitioning, or adjacency relations in later sections).

**2.1.4 Context indices and ladder notation**

In addition to tick indices, we use a discrete **context index** to label bands on the ladder.

**Definition 2.1.4 (Context index set).**  
The set of **context indices** is  
[  
N := { n \mid n \in \mathbb{Z} },  
]  
with elements referred to as **bands** or **context levels**.

We adopt the convention:

* (n=0) is the **hinge band** (special role to be made precise later).
* (n<0) are **inner bands**.
* (n>0) are **outer bands**.

When convenient, we denote a generic band by (n \in N) and refer to the collection of all bands as the **context ladder**.

**2.1.5 Basic maps on band indices and radial analogues**

To relate discrete band indices to continuous profiles (e.g. dimension profiles), we will sometimes use a **radial parameter** (r) and a map between (n) and (r).

* Let (r) be a real or integer parameter used to describe context levels in a continuous or quasi-continuous way.
* We will write (r(n)) for a (not necessarily one-to-one) map from band indices to radial values, and sometimes (n(r)) for an inverse or approximate inverse.
* In many constructions, we assume a monotone relationship (inner bands mapped to smaller |r|, outer bands to larger |r|), but the exact form of (r(n)) is left unspecified unless needed for a particular derivation.

At this preliminary stage, it is enough to note that:

* Context levels are indexed discretely by (n),
* Yet many structural functions (such as dimension profiles) will be written as (D(r)) for convenience, with the understanding that (D(n) = D(r(n))) along the ladder.

**2.1.6 Summary**

Section 2.1 has fixed the following basic ingredients:

* A tick set (T) with integer labels and a simple separation map (\Delta).
* An additive monoid of tick differences (\mathbb{T}\_\Delta).
* Generic index sets (A) for families of PMSs and carriers.
* A context index set (N) for bands on the ladder, and a radial parameter (r) for continuous profiles.

These objects provide the minimal scaffolding for the next subsection, where we introduce the abstract state space (\mathcal{H}), define carriers (\mathcal{C}\_k) more formally, and begin to specify the domains of the primitive operators.

**2.2 State Space (\mathcal{H}) and Carriers (\mathcal{C}\_k)**

In Section 1 we introduced PMSs, IN/ON, and CSs at a conceptual level. Section 2.1 fixed the basic tick and context indices. We now introduce the **abstract state space** (\mathcal{H}) and give a precise definition of the **tick-state carrier** (\mathcal{C}\_k), which will be the main object on which the primitive operators act.

**2.2.1 State space and tick-states**

We first specify an abstract state space for tick-level descriptions.

**Definition 2.2.1 (State space).**  
The **state space** (\mathcal{H}) is an abstract set whose elements are called **tick-states**. An element (h \in \mathcal{H}) encodes the information about “where we are” at a tick, at a level suitable for the operator algebra.

At this stage we assume only that:

* (\mathcal{H}) is a non-empty set.
* No particular additional structure (linear, metric, measure) is imposed unless explicitly introduced later (for example, when amplitudes or inner products are required).

When we need to refer to the state at a specific tick (k), we write (h\_k \in \mathcal{H}).

**Definition 2.2.2 (State–tick pair).**  
A **state–tick pair** is a pair  
[  
(k, h\_k), \quad k \in T,; h\_k \in \mathcal{H},  
]  
which records a tick index together with the corresponding tick-state.

The state–tick pair does not yet include the PMS data; it is a convenient intermediate object that allows us to separate “state-level” structure (in (\mathcal{H})) from PMS-level structure (IN/ON).

**2.2.2 Carriers as minimal tick descriptors**

We now refine the notion of carrier introduced informally in Section 1.2 and 1.3.

**Definition 2.2.3 (Tick-state carrier).**  
For each tick (k \in T), a **tick-state carrier** (or simply **carrier**) is a tuple  
[  
\mathcal{C}\_k := (k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k),  
]  
where:

* (k \in T) is the tick label,
* (h\_k \in \mathcal{H}) is the tick-state,
* (\mathrm{PMS}\_k = (\mathrm{IN}\_k, \mathrm{ON}\_k)) is the Present-Moment Sphere at tick (k).

We interpret (\mathcal{C}\_k) as the **minimal complete datum** needed at tick (k) for the purposes of:

* applying any of the primitive operators, and
* determining how ledger quantities and flip counts update from this tick to the next.

In other words:

If we know (\mathcal{C}\_k), we have everything the theory needs in order to describe the next admissible step in the evolution.

**2.2.3 Admissibility predicates**

Not every arbitrary combination of ((k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k)) is allowed. We introduce abstract **admissibility relations** to constrain which carriers are considered consistent.

**Definition 2.2.4 (Admissible carriers).**  
For each tick (k), we define:

* A relation (R\_{\mathrm{IN/ON},k} \subseteq \mathrm{IN}\_k \times \mathrm{ON}\_k), encoding which pairs of record and potential are mutually compatible at tick (k).
* A relation (R\_{\mathrm{state},k} \subseteq \mathcal{H} \times (\mathrm{IN}\_k \times \mathrm{ON}\_k)), encoding which combinations ((h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k)) are jointly consistent.

A carrier (\mathcal{C}\_k = (k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k)) is called **admissible** if:

* ((\mathrm{IN}\_k, \mathrm{ON}*k)) satisfies the IN/ON compatibility conditions (i.e. belongs to the appropriate domain for (R*{\mathrm{IN/ON},k})), and
* ((h\_k, \mathrm{IN}\_k, \mathrm{ON}*k)) lies in the state–PMS relation (R*{\mathrm{state},k}).

We do not specify these relations explicitly; instead:

* They are implicitly constrained by:
  + the operator algebra (which operators can act where),
  + the ledger invariants (e.g. record and exposure budgets),
  + and the fractal structure of IN (e.g. dimension and basins).

We require only that:

* If (\mathcal{C}\_k) is admissible and a primitive operator is defined on (\mathcal{C}*k), then the resulting (\mathcal{C}*{k'}) is also admissible.
* Admissibility is preserved along any allowed operator path.

**2.2.4 Operators as maps on carriers**

With carriers defined, we can state the general form of operator action without yet specifying the detailed rules for each operator.

**Definition 2.2.5 (Operator action on carriers).**  
For each primitive operator (O \in {F,S,T,C,CT}), there exists a domain (\mathcal{D}\_O) of admissible carriers such that  
[  
O : \mathcal{D}*O \subseteq {\mathcal{C}k \mid k \in T} \to {\mathcal{C}{k'} \mid k' \in T},  
]  
with  
[  
O\bigl( (k, h\_k, \mathrm{IN}k, \mathrm{ON}k) \bigr)  
= (k', h{k'}, \mathrm{IN}{k'}, \mathrm{ON}*{k'}).  
]

We impose two generic constraints:

1. **Tick progression.**  
   If (O(\mathcal{C}*k) = \mathcal{C}*{k'}), then (k' \geq k). Often we take “one-step” operators with (k' = k+1), but the framework allows multi-tick moves in principle.
2. **Admissibility preservation.**  
   If (\mathcal{C}\_k) is admissible and (O) is defined on (\mathcal{C}*k), then (\mathcal{C}*{k'} = O(\mathcal{C}\_k)) is also admissible.

We do **not** yet specify how each operator transforms (h\_k), (\mathrm{IN}\_k), and (\mathrm{ON}\_k); this is the role of later sections (particularly the operator algebra in Part III). Here we are only fixing the type and general shape of operator action.

**2.2.5 Operator words and carrier paths**

Given the primitive operators, we define **operator words** and the paths they induce on carriers.

**Definition 2.2.6 (Operator words).**  
Let  
[  
\mathcal{O} := {F,S,T,C,CT}  
]  
be the alphabet of primitive operators. A **word** in this alphabet is a finite sequence  
[  
\Pi = O\_r \circ O\_{r-1} \circ \dotsm \circ O\_1,\quad O\_i \in \mathcal{O},  
]  
written in the order they are applied.

If (\mathcal{C}*{k\_0}) is an admissible carrier and all intermediate compositions are defined and admissible, then we write  
[  
\Pi(\mathcal{C}*{k\_0}) = \mathcal{C}\_{k\_r}  
]  
for the resulting carrier after the full word (\Pi) is applied.

We call the sequence  
[  
\mathcal{C}*{k\_0} \xrightarrow{O\_1} \mathcal{C}*{k\_1} \xrightarrow{O\_2} \dots \xrightarrow{O\_r} \mathcal{C}\_{k\_r}  
]  
an **operator path** or **carrier path** generated by (\Pi).

This carrier path is the basic object from which:

* flip events and flip-count vectors will be defined (next subsection), and
* coarse-grained quantities like (\Delta t, \Delta \tau, |\Delta x|) and ledger changes will be computed (later sections).

**2.2.6 Reachability and carrier classes**

The notion of **reachability** organizes carriers into classes defined by which paths exist between them.

**Definition 2.2.7 (Reachability).**  
Let (\mathcal{C}*{k\_0}) be an admissible carrier. A carrier (\mathcal{C}*{k\_r}) is said to be **reachable** from (\mathcal{C}*{k\_0}) if there exists at least one admissible operator word (\Pi) such that  
[  
\mathcal{C}*{k\_r} = \Pi(\mathcal{C}\_{k\_0}),  
]  
with all intermediate carriers along the path admissible.

The set of all carriers reachable from (\mathcal{C}*{k\_0}) is the* ***reachability class*** *[  
\mathcal{R}(\mathcal{C}*{k\_0}) := { \mathcal{C}*{k\_r} \mid \exists, \Pi \text{ admissible with } \Pi(\mathcal{C}*{k\_0}) = \mathcal{C}\_{k\_r} }.  
]

Later, we will refine this with equivalence relations that identify carriers which differ only by “neutral moves” that do not change certain derived quantities (flip counts, ledger values, interval). For now, the reachability class simply encodes which carriers can be connected by some admissible evolution.

**2.2.7 Summary**

In Section 2.2 we have:

* Defined the state space (\mathcal{H}) and tick-states (h\_k).
* Defined carriers (\mathcal{C}\_k = (k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k)) as the minimal tick-level objects on which operators act.
* Introduced abstract admissibility relations that constrain which carriers are allowed.
* Specified the general form of operator action on carriers and the notion of operator paths and reachability classes.

These constructions provide the bridge between the conceptual PMS/IN/ON picture in Part I and the operator algebra that will be developed in Part III. The next subsection (2.3) will introduce the primitive operator set more explicitly and set up the language for flip words, flip events, and flip-count vectors.

**2.3 Primitive Operator Set**

The primitive operators are the basic “moves” that update carriers from one tick to another. They do not act on an external spacetime; instead, they directly transform the PMS parts (IN/ON) and the abstract state (h\_k) inside the carrier. All higher-level structures (interval, arrow of time, fields, etc.) will ultimately be built from compositions of these operators.

In this section we:

* Fix the **alphabet** of primitive operators.
* Specify their **domains** and **codomains** at the level of carriers.
* State the **generic form** of their action, without yet giving the full ledger or interval rules (that comes later in Part III).

**2.3.1 Operator alphabet**

We collect the primitive operators into a finite set:

**Definition 2.3.1 (Primitive operator set).**  
The set of primitive operators is  
[  
\mathcal{O} := { F, S, T, C, CT },  
]  
where:

* (F) is the **Renew** operator.
* (S) is the **Sink** operator.
* (T) is the **Trade** operator.
* (C) is the **Sync** (Collective Sphere) operator.
* (CT) is the **Framing** operator, understood as a composite of (C) and (T) in a specific order (to be fixed below).

Each element of (\mathcal{O}) is treated as a primitive symbol when forming operator words; the fact that (CT) is conceptually a composite does not change its status as a primitive *letter* in the algebra of words.

**2.3.2 Domains and codomains**

Each operator acts on carriers and returns a new carrier. We keep the most general form here and defer detailed update rules to later sections.

**Definition 2.3.2 (Operator domains).**  
For each (O \in \mathcal{O}), there is a domain (\mathcal{D}\_O) consisting of admissible carriers:  
[  
\mathcal{D}\_O \subseteq { \mathcal{C}*k \mid k \in T },  
]  
such that  
[  
O : \mathcal{D}O \to { \mathcal{C}{k'} \mid k' \in T }, \qquad  
\mathcal{C}*{k'} = O(\mathcal{C}\_k).  
]

We impose two generic constraints:

1. **Tick monotonicity**  
   If (\mathcal{C}\_{k'} = O(\mathcal{C}\_k)), then  
   [  
   k' \geq k,  
   ]  
   with the most common case being (k' = k+1) (a one-tick update).
2. **Admissibility preservation**  
   If (\mathcal{C}\_k) is admissible and (O) is defined on (\mathcal{C}*k), then (\mathcal{C}*{k'}) is also admissible.

These conditions ensure that operator paths can be composed without leaving the space of allowable carriers.

**2.3.3 Renew operator (F)**

Informally, the Renew operator (F) is responsible for **refreshing or enlarging the potential** side of the PMS, possibly in response to changes in outer or boundary conditions.

**Definition 2.3.3 (Renew operator).**  
Let (\mathcal{C}\_k = (k, h\_k, \mathrm{IN}*k, \mathrm{ON}k)) be an admissible carrier in (\mathcal{D}F). Then  
[  
F(\mathcal{C}k) = \mathcal{C}{k'} = (k', h{k'}, \mathrm{IN}{k'}, \mathrm{ON}*{k'}),  
]  
with:

* (k' \geq k) (typically (k' = k+1)),
* (\mathrm{IN}\_{k'}) equal to or compatible with (\mathrm{IN}\_k) (record is not erased by pure renewal),
* (\mathrm{ON}\_{k'}) updated to represent a new or expanded set of admissible potentials,
* (h\_{k'}) updated by an induced operator (\hat{F}) on the abstract state space (\mathcal{H}).

At this level we do not specify the detailed form of (\hat{F}) or the precise change in (\mathrm{ON}\_k); later sections will impose ledger and dimension constraints that shape what counts as an allowed renewal.

Intuitively:

* (F) prepares **new “branches”** in (\mathrm{ON}) consistent with the existing record in (\mathrm{IN}).

**2.3.4 Sink operator (S)**

The Sink operator (S) implements the **commitment of potential into record**: structures that were previously admissible in (\mathrm{ON}) are absorbed into (\mathrm{IN}).

**Definition 2.3.4 (Sink operator).**  
Let (\mathcal{C}\_k = (k, h\_k, \mathrm{IN}*k, \mathrm{ON}k)) be an admissible carrier in (\mathcal{D}S). Then  
[  
S(\mathcal{C}k) = \mathcal{C}{k'} = (k', h{k'}, \mathrm{IN}{k'}, \mathrm{ON}*{k'}),  
]  
with:

* (k' \geq k) (typically (k' = k+1)),
* (\mathrm{IN}\_{k'}) containing (\mathrm{IN}\_k) plus additional structure drawn from what was admissible in (\mathrm{ON}\_k),
* (\mathrm{ON}\_{k'}) representing a **reduced** or otherwise updated set of admissible potentials, consistent with the new record,
* (h\_{k'}) updated by an induced operator (\hat{S}) on (\mathcal{H}).

In later sections, Sink will be tied to monotonicity of the record budget (I(\mathcal{C}\_k)) and to the formation of “plateaus” in the IN structure, but here we only fix its role as **record-growing** and **potential-restricting**.

**2.3.5 Trade operator (T)**

The Trade operator (T) is responsible for **boundary exchanges** between IN and ON, allowing record to reorganize or shift without a net loss/gain of total capacity.

**Definition 2.3.5 (Trade operator).**  
Let (\mathcal{C}\_k = (k, h\_k, \mathrm{IN}*k, \mathrm{ON}k)) be an admissible carrier in (\mathcal{D}T). Then  
[  
T(\mathcal{C}k) = \mathcal{C}{k'} = (k', h{k'}, \mathrm{IN}{k'}, \mathrm{ON}*{k'}),  
]  
with:

* (k' \geq k) (again, typically (k' = k+1)),
* (\mathrm{IN}*{k'}) and (\mathrm{ON}*{k'}) obtained by **reallocating** elements across the boundary between record and potential, subject to ledger constraints,
* The total capacity (record plus exposure) preserved, so that later we can write  
  [  
  K(\mathcal{C}\_{k'}) = K(\mathcal{C}\_k).  
  ]
* (h\_{k'}) updated by a corresponding operator (\hat{T}) on (\mathcal{H}).

Conceptually, (T) allows “what counts as record” and “what remains potential” to be rearranged while preserving an overall budget.

**2.3.6 Sync operator (C)**

The Sync operator (C) acts to establish or maintain a **Collective Sphere** structure by aligning PMS boundaries across multiple carriers.

Formally, (C) can be viewed either:

* as an operator on individual carriers (\mathcal{C}\_k) that promotes alignment with a CS, or
* as an operator on CS carriers (\mathcal{C}\_k^{\mathrm{CS}}(A)).

Here we specify the single-carrier form.

**Definition 2.3.6 (Sync operator).**  
Let (\mathcal{C}\_k = (k, h\_k, \mathrm{IN}*k, \mathrm{ON}k)) be an admissible carrier in (\mathcal{D}C). Then  
[  
C(\mathcal{C}k) = \mathcal{C}{k'} = (k', h{k'}, \mathrm{IN}{k'}, \mathrm{ON}*{k'}),  
]  
such that:

* (k' \geq k),
* (\mathrm{IN}*{k'}), (\mathrm{ON}*{k'}) are adjusted to become compatible with a **shared boundary** description (B\_{k'}) for some CS,
* (h\_{k'}) is mapped by (\hat{C}) into a state that reflects this synchronization (e.g. by enforcing common constraints across a set of carriers).

We will later assume that (C) is **idempotent** in the sense that applying it twice in a regime where synchronization is already achieved has no additional effect beyond the first application.

**2.3.7 Framing operator (CT)**

The Framing operator (CT) is the composite that plays a central role in later discussions of measurement and decoherence. In this volume, we treat it as a primitive symbol but note its intended structure.

**Definition 2.3.7 (Framing operator).**  
The Framing operator is the composition  
[  
CT := C \circ T,  
]  
applied in regimes where both (T) and (C) are defined and admissible. For a carrier (\mathcal{C}*k) in the joint domain (\mathcal{D}*{CT}), we write  
[  
CT(\mathcal{C}\_k) = C\bigl( T(\mathcal{C}\_k) \bigr).  
]

Informally:

* (T) first **reallocates** structure across the IN/ON boundary.
* (C) then **synchronizes** the resulting configuration across a CS.

Depending on parameter choices (not specified in detail here), (CT) can implement:

* **Strong framing**, in which alternative possibilities in (\mathrm{ON}) are effectively collapsed into a single shared record configuration across the CS.
* **Weak framing**, in which (\mathrm{ON}) is biased or partially reduced but not uniquely fixed.

In this core theory volume, we keep the description at the level of the operator and its domain/codomain; the detailed phenomenological reading of strong vs weak framing is relegated to interpretive and evidence-oriented texts.

**2.3.8 Summary**

Section 2.3 has:

* Fixed the **primitive operator alphabet** (\mathcal{O} = {F,S,T,C,CT}).
* Described, at the level of carriers, how each operator transforms ((k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k)) in a general way.
* Prepared the ground for:
  + the **ledger** definitions that track record and exposure budgets under these operators, and
  + the **flip-count invariants** and interval construction that use words in (\mathcal{O}).

The next subsection will make operator words and flip counts explicit, and then we will introduce the ledger and capacity functions that constrain admissible evolutions and give the algebra its time-ordered structure.

**2.4 Flip Words, Flip Counts, Neutral Moves**

This section formalizes **operator words** built from the primitive set (\mathcal{O} = {F,S,T,C,CT}), defines the associated **flip-count vectors**, and introduces **neutral words** and the induced equivalence on carrier paths. These objects will be used later to define invariants (interval, ledger relations) that depend only on flip counts, not on the detailed ordering of neutral moves.

**2.4.1 Operator words and length**

We start from the primitive alphabet (\mathcal{O} = {F,S,T,C,CT}).

**Definition 2.4.1 (Operator word).**  
An **operator word** (or **flip word**) is a finite sequence  
[  
\Pi = O\_m O\_{m-1} \dotsm O\_1,  
]  
where each (O\_j \in \mathcal{O}). We will usually interpret (\Pi) as the composition  
[  
\Pi := O\_m \circ O\_{m-1} \circ \dotsm \circ O\_1.  
]

The **length** of (\Pi) is  
[  
|\Pi| := m.  
]

If (\mathcal{C}*{k\_0}) is an admissible carrier such that each partial composition  
[  
\mathcal{C}*{k\_j} := O\_j \circ \dotsm \circ O\_1(\mathcal{C}*{k\_0})  
]  
is defined and admissible, we write  
[  
\Pi(\mathcal{C}*{k\_0}) = \mathcal{C}*{k\_m},  
]  
and call  
[  
\mathcal{C}*{k\_0} \xrightarrow{O\_1} \mathcal{C}*{k\_1} \xrightarrow{O\_2} \dots \xrightarrow{O\_m} \mathcal{C}*{k\_m}  
]  
the **carrier path** induced by (\Pi).

**2.4.2 Flip-count vectors**

For each word (\Pi) we define a vector that counts how many times each primitive occurs.

**Definition 2.4.2 (Flip-count vector).**  
Given an operator word (\Pi), its **flip-count vector** is  
[  
\nu(\Pi) := (\nu\_F(\Pi), \nu\_S(\Pi), \nu\_T(\Pi), \nu\_C(\Pi), \nu\_{CT}(\Pi)) \in \mathbb{N}^5,  
]  
where:

* (\nu\_F(\Pi)) = number of occurrences of (F) in (\Pi),
* (\nu\_S(\Pi)) = number of occurrences of (S) in (\Pi),
* (\nu\_T(\Pi)) = number of occurrences of (T) in (\Pi),
* (\nu\_C(\Pi)) = number of occurrences of (C) in (\Pi),
* (\nu\_{CT}(\Pi)) = number of occurrences of (CT) in (\Pi).

This is well-defined for any word in the free monoid generated by (\mathcal{O}), independent of admissibility on a particular carrier.

We will also write  
[  
\nu(\Pi) = (\nu\_O(\Pi))\_{O \in \mathcal{O}}  
]  
when it is convenient to index the components by the operators themselves.

**Lemma 2.4.3 (Additivity under concatenation).**  
Let (\Pi\_1, \Pi\_2) be operator words and define their concatenation  
[  
\Pi := \Pi\_2 \Pi\_1.  
]  
Then  
[  
\nu(\Pi) = \nu(\Pi\_2) + \nu(\Pi\_1)  
]  
(component-wise addition in (\mathbb{N}^5)).

*Proof.* Immediate from the definition of (\nu) as counts of occurrences. (\square)

**2.4.3 Neutral words**

Some compositions of operators do not change any of the quantities that later appear in invariants (interval, ledger values, etc.). We abstract these as **neutral words**.

**Definition 2.4.4 (Neutral word, syntactic notion).**  
A word (\Pi) is **syntactically neutral** if  
[  
\nu(\Pi) = 0,  
]  
i.e. none of the primitive symbols (F,S,T,C,CT) appears in (\Pi). In practice this means (\Pi) is the empty word, which we denote by (\epsilon).

Because we allow only the primitive alphabet (\mathcal{O}), the only syntactically neutral word is (\epsilon). However, in the presence of additional derived operations or when we consider words that act trivially on carriers, a more general notion is useful.

**Definition 2.4.5 (Neutral word, action-level notion).**  
Let (\mathcal{S}) be a class of carriers (typically all carriers reachable from some initial carrier (\mathcal{C}\_0)). A word (\Pi) is **neutral on (\mathcal{S})** if for all (\mathcal{C} \in \mathcal{S}) where (\Pi(\mathcal{C})) is defined and admissible, we have  
[  
\Pi(\mathcal{C}) = \mathcal{C}.  
]

In other words, (\Pi) acts as the identity on (\mathcal{S}). Such a word may have non-zero flip counts if its action cancels at the carrier level (for example, if some inverses or approximate inverses exist in specific regimes); we do not require a detailed classification of these words in this volume. What matters is that there exists a class of words that **do not change** the carriers in (\mathcal{S}).

When we speak of **neutral moves** along a carrier path, we mean compositions corresponding to such action-level neutral words.

**2.4.4 Equivalence of words and paths**

For many constructions (interval, ledger invariants), only the flip-count vector matters, not the specific order of non-neutral moves or insertion of neutral moves. This motivates an equivalence relation on words and the induced equivalence on carrier paths.

**Definition 2.4.6 (Word equivalence by flip counts).**  
Two words (\Pi\_1) and (\Pi\_2) are **flip-count equivalent**, written (\Pi\_1 \sim\_\nu \Pi\_2), if  
[  
\nu(\Pi\_1) = \nu(\Pi\_2).  
]

This is an equivalence relation on the free monoid generated by (\mathcal{O}).

Given a fixed initial carrier (\mathcal{C}\_{k\_0}), we can refine this as follows.

**Definition 2.4.7 (Path equivalence modulo neutral moves).**  
Let (\mathcal{C}*{k\_0}) be an admissible carrier. Two words (\Pi\_1,\Pi\_2) are \*\*path-equivalent at (\mathcal{C}*{k\_0})\*\*, written (\Pi\_1 \approx\_{\mathcal{C}\_{k\_0}} \Pi\_2), if:

1. Both (\Pi\_1(\mathcal{C}*{k\_0})) and (\Pi\_2(\mathcal{C}*{k\_0})) are defined and admissible; and
2. There exists a neutral word (\Pi\_0) (neutral on the reachability class of (\mathcal{C}\_{k\_0})) such that  
   [  
   \Pi\_2 = \Pi\_0 \Pi\_1  
   ]  
   or  
   [  
   \Pi\_1 = \Pi\_0 \Pi\_2  
   ]  
   as compositions (where defined).

Intuitively, (\Pi\_1) and (\Pi\_2) differ only by insertion/removal of neutral moves along the path from (\mathcal{C}\_{k\_0}).

In many applications we will restrict attention to **flip-count classes**: sets of paths that share the same (\nu) and differ only by neutral moves. For any initial carrier (\mathcal{C}*{k\_0}) and any given flip-count vector (\nu), we can consider the set of all paths out of (\mathcal{C}*{k\_0}) whose generating words have that flip-count and are considered equivalent modulo neutral moves.

**2.4.5 Reachability classes revisited**

Recall from Definition 2.2.7 that the reachability class (\mathcal{R}(\mathcal{C}*{k\_0})) consists of all carriers reachable from (\mathcal{C}*{k\_0}) by some admissible word. We refine this with flip counts.

**Definition 2.4.8 (Reachability by flip-count class).**  
For a fixed initial carrier (\mathcal{C}*{k\_0}) and a flip-count vector (\nu \in \mathbb{N}^5), define  
[  
\mathcal{R}*\nu(\mathcal{C}*{k\_0}) := \left{ \mathcal{C}*{k\_r} ,\middle|,  
\exists \Pi \text{ admissible with } \Pi(\mathcal{C}*{k\_0}) = \mathcal{C}*{k\_r}  
\text{ and } \nu(\Pi) = \nu  
\right}.  
]

Thus, (\mathcal{R}*\nu(\mathcal{C}*{k\_0})) collects all carriers reachable with **exactly** the flip-count vector (\nu), irrespective of the internal ordering and neutral insertions that may appear in the generating paths.

Later, when we define:

* corresponding increments (\Delta t(\nu), \Delta \tau(\nu), |\Delta x(\nu)|), and
* ledger changes attached to flip-count classes,

these (\mathcal{R}\_\nu) sets will be the natural domains on which the invariant relationships are imposed.

**2.4.6 Summary**

Section 2.4 has:

* Defined operator words and their lengths.
* Introduced the flip-count vector (\nu(\Pi)) and its additivity under concatenation.
* Defined neutral words (syntactic and action-level) and used them to formulate path equivalence modulo neutral moves.
* Refined the notion of reachability into flip-count classes (\mathcal{R}*\nu(\mathcal{C}*{k\_0})).

In the next section we will introduce the ledger and capacity functions (I(\mathcal{C}), E(\mathcal{C}), K(\mathcal{C})) and state the basic monotonicity and conservation properties that these functions obey under the action of the primitive operators. These ledger properties, together with the flip-count structure defined here, will underpin the construction of an invariant interval and an intrinsic arrow of time.

**3. Tick-Operator Algebra, Arrow, and Interval**

**3.1 State Space & Carriers (Formal Assumptions)**

Parts 1 and 2 introduced PMSs, IN/ON, CSs, the state space (\mathcal{H}), and carriers (\mathcal{C}\_k), together with the primitive operators and flip words. In this section we collect and slightly tighten the assumptions about (\mathcal{H}), carriers, and admissibility that will be used explicitly in the algebraic results that follow (ledger laws, arrow of time, and invariant interval).

The aim here is not to introduce new objects, but to make explicit **which properties are taken as axioms** in the operator-algebra layer.

**3.1.1 Structural assumptions on (\mathcal{H})**

Recall that (\mathcal{H}) is the abstract state space for tick-states (h\_k). For the purposes of the algebra, we assume:

**Axiom 3.1.1 (State space).**  
The state space (\mathcal{H}) is a non-empty set equipped with:

1. A distinguished subset (\mathcal{H}*{\mathrm{adm}} \subseteq \mathcal{H}) of* ***admissible states****, such that all tick-states (h\_k) that appear in carriers are elements of (\mathcal{H}*{\mathrm{adm}}).
2. For each primitive operator (O \in {F,S,T,C,CT}), a (possibly partial) map  
   [  
   \hat{O} : \mathcal{H}*{\mathrm{adm}} \to \mathcal{H}*{\mathrm{adm}},  
   ]  
   which represents the induced action of (O) on the abstract state.

We do **not** assume linearity, a norm, or an inner product on (\mathcal{H}) at this stage. Where such structure is needed (e.g., for amplitude-like descriptions or Born-style rules), it will be introduced explicitly later and only on the relevant subspaces.

**3.1.2 Carriers and admissibility**

We now restate carriers and admissibility in the form we will use to formulate algebraic laws.

**Definition 3.1.2 (Carrier).**  
A **carrier** is a tuple  
[  
\mathcal{C} = (k, h, \mathrm{IN}, \mathrm{ON}),  
]  
where:

* (k \in T) is a tick label,
* (h \in \mathcal{H}\_{\mathrm{adm}}) is an admissible state,
* ((\mathrm{IN},\mathrm{ON})) is a pair of structures that form a PMS.

When we wish to emphasize the tick index, we write (\mathcal{C}\_k) instead of (\mathcal{C}), with the understanding that  
[  
\mathcal{C}\_k = (k, h\_k, \mathrm{IN}\_k, \mathrm{ON}\_k).  
]

**Axiom 3.1.3 (Admissible carriers).**  
There exists a distinguished subset (\mathcal{C}\_{\mathrm{adm}}) of carriers, called **admissible carriers**, with the following properties:

1. For each admissible carrier (\mathcal{C} = (k,h,\mathrm{IN},\mathrm{ON}) \in \mathcal{C}\_{\mathrm{adm}}):
   * ((\mathrm{IN},\mathrm{ON})) obeys the IN/ON role separation and completeness conditions stated in Section 1.3.
   * (h) is compatible with ((\mathrm{IN},\mathrm{ON})) in the sense that it can be obtained from them via some (possibly many-to-one) map (\Phi):  
     [  
     h = \Phi(\mathrm{IN},\mathrm{ON}).  
     ]
2. If (\mathcal{C} \in \mathcal{C}*{\mathrm{adm}}) and a primitive operator (O) is defined on (\mathcal{C}), then the result (O(\mathcal{C})) is also in (\mathcal{C}*{\mathrm{adm}}).
3. For every (\mathcal{C} \in \mathcal{C}*{\mathrm{adm}}), the pair ((h,\mathrm{IN},\mathrm{ON})) lies in the compatibility relation (R*{\mathrm{state},k}) introduced in Section 2.2.

These conditions guarantee that the evolution generated by primitive operators never leaves the space of structurally well-formed carriers.

**3.1.3 Operator action re-stated**

With (\mathcal{H}*{\mathrm{adm}}) and (\mathcal{C}*{\mathrm{adm}}) in place, we can summarize operator action in a compact form that we will use repeatedly.

**Axiom 3.1.4 (Primitive operator action).**  
For each (O \in {F,S,T,C,CT}) there is a domain (\mathcal{D}*O \subseteq \mathcal{C}*{\mathrm{adm}}) such that:

1. (O : \mathcal{D}*O \to \mathcal{C}*{\mathrm{adm}}),
2. If  
   [  
   O(k,h,\mathrm{IN},\mathrm{ON}) = (k',h',\mathrm{IN}',\mathrm{ON}'),  
   ]  
   then  
   [  
   k' \geq k  
   ]  
   (tick monotonicity),
3. The abstract state transforms by the corresponding state-operator:  
   [  
   h' = \hat{O}(h).  
   ]

We do not yet specify the detailed transformation rules for (\mathrm{IN}) and (\mathrm{ON}); those will be constrained by the ledger properties and other algebraic laws in the next subsections. The important point is that **every primitive operator has a well-defined action on carriers and on the abstract state**, with tick labels non-decreasing along admissible paths.

**3.1.4 Operator paths revisited**

Given these assumptions, we reiterate the notion of operator paths in a form that prepares for ledger and interval constructions.

Let (\mathcal{C}*0 \in \mathcal{C}*{\mathrm{adm}}) be an initial carrier. A word  
[  
\Pi = O\_m \dotsm O\_1  
]  
with each (O\_j \in \mathcal{O}) and each intermediate carrier admissible yields a sequence  
[  
\mathcal{C}\_0  
\xrightarrow{O\_1}  
\mathcal{C}\_1  
\xrightarrow{O\_2}  
\dots  
\xrightarrow{O\_m}  
\mathcal{C}\_m,  
]  
where (\mathcal{C}*j \in \mathcal{C}*{\mathrm{adm}}) for all (j), and the tick indices satisfy  
[  
k\_0 \le k\_1 \le \dots \le k\_m.  
]

Correspondingly, the abstract states satisfy  
[  
h\_j = \hat{O}*j(h*{j-1})  
]  
for (j=1,\dots,m).

This path structure is the backdrop against which we will:

* track ledger quantities (I(\mathcal{C}\_j), E(\mathcal{C}\_j), K(\mathcal{C}\_j)),
* define the arrow of time via monotonic changes in these quantities, and
* define interval components (\Delta t, \Delta \tau, |\Delta x|) as functions of the flip-count vector (\nu(\Pi)).

**3.1.5 Reachability and flip-count classes as working domains**

The reachability and flip-count constructions from Section 2.4 now acquire a more explicit role.

Given an initial carrier (\mathcal{C}*0 \in \mathcal{C}*{\mathrm{adm}}):

* The **reachability class** (\mathcal{R}(\mathcal{C}\_0)) is the set of all carriers reachable by some admissible operator path.
* For any flip-count vector (\nu), the subset (\mathcal{R}\_\nu(\mathcal{C}\_0)) consists of carriers reachable via paths with flip-count (\nu).

In what follows we will:

* Define derived quantities (ledger, interval components) as functions of (\nu).
* Assert that these quantities are **invariants** over all carriers in a given (\mathcal{R}\_\nu(\mathcal{C}\_0)), up to neutral moves.

This is the core idea: **the physically relevant quantities are attached to flip-count classes, not to specific detailed words or paths**.

**3.1.6 Summary**

Section 3.1 collects the formal assumptions that underlie the operator algebra:

* (\mathcal{H}) has a distinguished admissible subset (\mathcal{H}\_{\mathrm{adm}}) and carries induced state-operators (\hat{O}).
* Carriers (\mathcal{C} = (k,h,\mathrm{IN},\mathrm{ON})) form an admissible set (\mathcal{C}\_{\mathrm{adm}}) that is closed under primitive operator action.
* Operator paths through (\mathcal{C}\_{\mathrm{adm}}) are the basic histories from which ledger and interval structures are defined.
* Flip-count classes (\mathcal{R}\_\nu(\mathcal{C}\_0)) provide the natural domains for invariants.

With these assumptions in place, we can now introduce the **ledger and capacity functions** and state the algebraic laws that govern them. Those ledger laws will encode the arrow of time, and together with flip-count invariants they will lead to the construction of a Lorentz-like interval in Section 3.4.

**3.2 Ledger Functions and Capacity**

The **ledger** attaches scalar “budgets” to carriers, tracking how much of the available structure has become **record** and how much remains as **potential**. These budgets are used to:

* express an **intrinsic arrow of time** (via monotonicity), and
* support the construction of an **invariant interval** (via relations between ledger changes and flip counts).

This section:

1. Defines the ledger functions (I, E, K) on carriers;
2. States basic constraints (non-negativity, capacity); and
3. Specifies how each primitive operator affects (I) and (E) at the single-step level.

**3.2.1 Ledger definitions and capacity constraint**

We assign to each admissible carrier a pair of non-negative budgets and a capacity.

**Definition 3.2.1 (Ledger functions).**  
Let (\mathcal{C}) be an admissible carrier. The **ledger** consists of three functions:

* (I : \mathcal{C}*{\mathrm{adm}} \to \mathbb{R}*{\ge 0}), the **record budget**,
* (E : \mathcal{C}*{\mathrm{adm}} \to \mathbb{R}*{\ge 0}), the **exposure budget**,
* (K : \mathcal{C}*{\mathrm{adm}} \to \mathbb{R}*{> 0}), the **capacity**,

such that for every admissible carrier (\mathcal{C}),  
[  
I(\mathcal{C}) + E(\mathcal{C}) = K(\mathcal{C}).  
]

We will often abbreviate, for a carrier (\mathcal{C}\_k),

* (I(\mathcal{C}\_k)) as (I\_k),
* (E(\mathcal{C}\_k)) as (E\_k),
* (K(\mathcal{C}\_k)) as (K\_k).

Thus,  
[  
I\_k + E\_k = K\_k.  
]

**Axiom 3.2.2 (Non-negativity and finite capacity).**  
For all admissible carriers (\mathcal{C}),  
[  
0 \le I(\mathcal{C}) \le K(\mathcal{C}),\quad  
0 \le E(\mathcal{C}) \le K(\mathcal{C}).  
]

So the ledger expresses a **finite budget** (K(\mathcal{C})), partitioned between record and potential.

**Axiom 3.2.3 (Local constancy of capacity on subsystems).**  
On a given subsystem or reachability class (as specified by context), we often work in regimes where  
[  
K(\mathcal{C}\_k) \equiv K  
]  
for all carriers (\mathcal{C}\_k) in that class. When (K) varies, its variation is treated explicitly; in either case, capacity conservation under primitive operators is imposed as a separate requirement (below).

Interpretation:

* (I(\mathcal{C})) measures how much of the available capacity is already part of **record** (IN-side commitment).
* (E(\mathcal{C})) measures how much remains as **potential** (ON-side availability).

**3.2.2 Capacity conservation**

We assume that the primitive operators do not create or destroy capacity; they only redistribute it between record and exposure, possibly together with changes in internal organization.

**Axiom 3.2.4 (Capacity conservation under primitives).**  
For each primitive operator (O \in {F,S,T,C,CT}) and any admissible carrier (\mathcal{C}) in its domain,  
[  
K\bigl(O(\mathcal{C})\bigr) = K(\mathcal{C}).  
]

In particular, for a one-step update (\mathcal{C}*k \mapsto \mathcal{C}*{k'} = O(\mathcal{C}*k)),  
[  
I*{k'} + E\_{k'} = I\_k + E\_k = K\_k = K\_{k'}.  
]

Thus, the ledger distinguishes **allocation** (how capacity is split into record and exposure) from the total **amount** of capacity, which is preserved by each tick-level primitive.

**3.2.3 Renew (F): exposing potential**

The Renew operator (F) is responsible for **exposing or refreshing potential** in the Outer Network, without erasing record.

Recall from Section 2.3 that for an admissible carrier (\mathcal{C}*k = (k,h\_k,\mathrm{IN}k,\mathrm{ON}k)) in the domain of (F), we have  
[  
F(\mathcal{C}k) = \mathcal{C}{k'} = (k',h{k'},\mathrm{IN}{k'},\mathrm{ON}*{k'}).  
]

We now state its ledger behavior.

**Axiom 3.2.5 (Ledger behavior of (F)).**  
For every admissible (\mathcal{C}*k) in the domain of (F), letting (\mathcal{C}*{k'} = F(\mathcal{C}\_k)),

* (F) does **not decrease record**:  
  [  
  I\_{k'} \ge I\_k.  
  ]  
  In many idealized contexts we take (I\_{k'} = I\_k) for a purely renewing step.
* (F) can **increase exposure** up to capacity:  
  [  
  E\_{k'} \ge E\_k,\quad E\_{k'} \le K\_{k'} - I\_{k'}.  
  ]
* Capacity is conserved:  
  [  
  I\_{k'} + E\_{k'} = K\_{k'} = K\_k.  
  ]

Interpretation: renewal **opens up** more potential in (\mathrm{ON}) consistent with the existing record, but it does not undo what has been committed to (\mathrm{IN}).

**3.2.4 Sink (S): committing potential to record**

The Sink operator (S) implements the **ON → IN flow**: potential becomes record.

For an admissible carrier (\mathcal{C}*k) in the domain of (S), we have  
[  
S(\mathcal{C}k) = \mathcal{C}{k'} = (k',h*{k'},\mathrm{IN}*{k'},\mathrm{ON}*{k'}).  
]

**Axiom 3.2.6 (Ledger behavior of (S)).**  
For every such step:

* (S) **increases record**:  
  [  
  I\_{k'} \ge I\_k.  
  ]
* (S) **decreases exposure**:  
  [  
  E\_{k'} \le E\_k.  
  ]
* Capacity is conserved:  
  [  
  I\_{k'} + E\_{k'} = K\_{k'} = K\_k.  
  ]

In a “pure” sink step, we can think schematically of  
[  
I\_{k'} = I\_k + \delta I,\quad E\_{k'} = E\_k - \delta I,\quad \delta I > 0,  
]  
with (\delta I) determined by how much ON content has been committed into IN.

Thus, (S) is a direct algebraic source of the **irreversible growth of record** and reduction of remaining potential along a path.

**3.2.5 Trade (T): boundary reallocation**

The Trade operator (T) rebalances structure across the IN/ON boundary without changing total capacity; it primarily reorganizes what counts as record versus potential.

For an admissible carrier (\mathcal{C}*k) in the domain of (T),  
[  
T(\mathcal{C}k) = \mathcal{C}{k'} = (k',h*{k'},\mathrm{IN}*{k'},\mathrm{ON}*{k'}).  
]

**Axiom 3.2.7 (Ledger behavior of (T)).**

* **Capacity preservation**:  
  [  
  I\_{k'} + E\_{k'} = K\_{k'} = K\_k.  
  ]
* In an idealized “pure trade” regime, (T) preserves record and exposure totals:  
  [  
  I\_{k'} = I\_k,\quad E\_{k'} = E\_k,  
  ]  
  while only changing the **internal organization** of (\mathrm{IN}) and (\mathrm{ON}).

More general versions of (T) may allow small shifts between (I) and (E), but in all cases:

* (T) does not violate the capacity constraint, and
* (T) does not itself introduce a systematic arrow of time; it is **boundary-balancing**, not inherently record-growing.

**3.2.6 Sync (C): synchronization and ledger neutrality**

The Sync operator (C) builds and maintains Collective Spheres by aligning PMS boundaries. At the ledger level, its action is essentially **neutral** in the sense that it does not drive record or exposure in a preferred direction.

For an admissible carrier (\mathcal{C}\_k) in the domain of (C),  
[  
C(\mathcal{C}*k) = \mathcal{C}*{k'}.  
]

**Axiom 3.2.8 (Ledger behavior of (C)).**

* Capacity is preserved:  
  [  
  I\_{k'} + E\_{k'} = K\_{k'} = K\_k.  
  ]
* In a synchronized regime, (C) is effectively **ledger-neutral**:  
  [  
  I\_{k'} = I\_k,\quad E\_{k'} = E\_k,  
  ]  
  with its primary effect being to enforce or maintain synchronization constraints across a CS, not to reallocate record or potential.

Additionally, we will later assume **idempotence** of (C) in fully synchronized contexts:  
[  
C(C(\mathcal{C})) = C(\mathcal{C}),  
]  
which is an algebraic statement independent of the ledger, but consistent with the idea that repeated synchronization has no further effect once alignment is achieved.

**3.2.7 Framing (CT): measurement-like commitment**

The Framing operator (CT = C \circ T) combines Trade and Sync and is used to model **framing/measurement-like processes**. At the ledger level, it behaves similarly to a sink when applied with sufficient strength, committing potential into record in a way that is shared across a CS.

For an admissible carrier (\mathcal{C}\_k) in the domain of (CT),  
[  
CT(\mathcal{C}*k) = \mathcal{C}*{k'}.  
]

**Axiom 3.2.9 (Ledger behavior of strong (CT)).**  
In regimes where (CT) is applied as a **strong framing** operation:

* Record does not decrease:  
  [  
  I\_{k'} \ge I\_k.  
  ]
* Exposure does not increase:  
  [  
  E\_{k'} \le E\_k.  
  ]
* Capacity is preserved:  
  [  
  I\_{k'} + E\_{k'} = K\_{k'} = K\_k.  
  ]

In many measurement contexts, the net effect is similar to a sink step followed by a synchronization across multiple PMSs: potential branches in (\mathrm{ON}) are reduced, and a specific record configuration is shared across a CS.

**3.2.8 Cumulative ledger behavior along paths**

For an operator word (\Pi = O\_m \dotsm O\_1) applied to an initial carrier (\mathcal{C}\_0),  
[  
\mathcal{C}\_0 \xrightarrow{O\_1} \mathcal{C}\_1 \xrightarrow{O\_2} \dotsb \xrightarrow{O\_m} \mathcal{C}\_m,  
]  
the single-step ledger rules combine to give cumulative inequalities. In particular, for any path that includes at least one sink or strong framing step, we typically have  
[  
I(\mathcal{C}\_m) \ge I(\mathcal{C}\_0),  
]  
with strict inequality whenever genuine ON → IN commitment occurs somewhere along the path.

Renew steps can expand exposure but do not erase record; trade and sync reorganize or align without violating capacity constraints. Thus, as ticks progress along admissible paths, **record tends to grow and uncommitted potential tends to shrink**, in a way that cannot be fully undone by the allowed operators.

This cumulative, one-way behavior of the ledger is the algebraic expression of the **time arrow**. In the next section (3.3), we will formalize this in terms of non-commutativity, idempotents, and global monotonicity theorems, and then connect it to the construction of an invariant interval.

**3.3 Algebraic Laws & Intrinsic Arrow**

This section collects a few simple but crucial algebraic properties of the primitive operators and ledger functions. Together, these encode an **intrinsic arrow of time**: along any admissible evolution, record cannot be erased in a way that restores both tick index and ledger values to their previous state.

We do not introduce new primitives here; we only organize and sharpen the consequences of the assumptions already stated.

**3.3.1 Composition and associativity on domains**

Let (\mathcal{O} = {F,S,T,C,CT}) be the primitive alphabet, and let (\mathcal{O}^\*) denote the set of finite words in this alphabet.

**Lemma 3.3.1 (Associativity of operator composition on carriers).  
Let (\Pi\_1,\Pi\_2,\Pi\_3 \in \mathcal{O}^\*), and let (\mathcal{C}) be an admissible carrier such that all intermediate compositions are defined and admissible. Then  
[  
\bigl( \Pi\_3 \circ (\Pi\_2 \circ \Pi\_1) \bigr)(\mathcal{C})**

\bigl( (\Pi\_3 \circ \Pi\_2) \circ \Pi\_1 \bigr)(\mathcal{C}).  
]

*Proof.* This is just associativity of function composition restricted to the admissible domains. (\square)

In practice, this means we can write long composites of primitive operators unambiguously, as long as we remain within the admissible subset of carriers.

**3.3.2 Non-commutation of renew and sink**

The key algebraic structure behind the arrow is the **non-commutation** of operators that grow record versus those that expose potential. We do not attempt to give a fully general commutation table; we focus on the case that matters for the ledger.

**Axiom 3.3.2 (Non-commutation of (F) and (S)).**  
There exists a non-empty class of admissible carriers (\mathcal{S} \subseteq \mathcal{C}\_{\mathrm{adm}}) such that for each (\mathcal{C} \in \mathcal{S}):

1. All four carriers  
   [  
   F(S(\mathcal{C})),\quad S(F(\mathcal{C})),\quad F(\mathcal{C}),\quad S(\mathcal{C})  
   ]  
   are defined and admissible.
2. At least one of the following non-commutation conditions holds:  
   [  
   F(S(\mathcal{C})) \neq S(F(\mathcal{C})),  
   ]  
   or  
   [  
   I(F(S(\mathcal{C}))) \neq I(S(F(\mathcal{C}))) \quad \text{or} \quad  
   E(F(S(\mathcal{C}))) \neq E(S(F(\mathcal{C}))).  
   ]

In other words, the order in which renewal and sinking are applied matters; the combination of “open up” (via (F)) and “commit” (via (S)) cannot be rearranged freely without altering the resulting carrier or its ledger.

This non-commutation is the algebraic origin of a directed structure: once a certain pattern of ON → IN flow has occurred, there is no inverse sequence of renew and sink steps that can restore the original carrier and its ledger values.

**3.3.3 Idempotent and semi-idempotent elements**

Some operators are idempotent or semi-idempotent in relevant regimes, which helps organize the algebra into “once-done, stays done” steps.

**Axiom 3.3.3 (Idempotence of (C) in synchronized regimes).**  
On any subset (\mathcal{S} \subseteq \mathcal{C}\_{\mathrm{adm}}) where carriers are fully synchronized with respect to a given CS (i.e., the synchronization constraint is already satisfied), we have  
[  
C(C(\mathcal{C})) = C(\mathcal{C})  
]  
for all (\mathcal{C} \in \mathcal{S}).

This reflects that once synchronization is achieved, further applications of (C) have no additional effect.

For sink-like and framing-like operations, we assume a weaker form of idempotence in the ledger sense:

**Axiom 3.3.4 (Ledger idempotence of (S) and strong (CT)).**  
For carriers in a regime where a given record configuration is already fully established, repeated applications of (S) or strong (CT) do not increase record further:  
[  
I(S(\mathcal{C})) = I(\mathcal{C}),\quad  
I(CT(\mathcal{C})) = I(\mathcal{C})  
]  
whenever (I(\mathcal{C})) has reached the relevant saturation bound for that regime.

These idempotent behaviors help define **plateaus** in record and synchronization, beyond which additional applications of the same operators do not change ledger values or synchronization properties.

**3.3.4 Monotonicity of record and exposure**

Collecting the single-step axioms for (F,S,T,C,CT), we can state global monotonicity behavior along any admissible path.

Let  
[  
\mathcal{C}\_0  
\xrightarrow{O\_1}  
\mathcal{C}\_1  
\xrightarrow{O\_2}  
\dots  
\xrightarrow{O\_m}  
\mathcal{C}\_m  
]  
be a carrier path generated by a word (\Pi = O\_m \dotsm O\_1).

Define the cumulative record and exposure differences  
[  
\Delta I := I(\mathcal{C}\_m) - I(\mathcal{C}\_0),\quad  
\Delta E := E(\mathcal{C}\_m) - E(\mathcal{C}\_0).  
]

**Proposition 3.3.5 (Global ledger bounds).**  
Along any admissible path as above,

1. (\Delta I \ge 0): record never decreases along a path that consists only of applications of (F,S,T,C,CT) satisfying the ledger axioms.
2. (\Delta E \le K(\mathcal{C}\_m) - I(\mathcal{C}\_m)): exposure cannot exceed the remaining capacity once record increases.
3. If at least one genuine sink or strong framing step occurs (i.e. at least one step with strict (I\_{k'} > I\_k)), then (\Delta I > 0).

*Sketch of proof.*

* (F) never decreases (I); in the pure-renew regime, (I) is unchanged.
* (S) and strong (CT) increase (I) and decrease (E).
* (T) and (C) are ledger-neutral in the idealized regime (preserving (I,E)).  
  Combining these stepwise, we see that (I) cannot decrease along a sequence of such steps, and strict increase occurs whenever a genuine ON → IN commitment happens. Capacity conservation gives the bound on (E). (\square)

This proposition encodes the **one-way growth of record** along operator paths.

**3.3.5 No non-trivial time-reversal in the primitive algebra**

The monotonicity of (I) combined with non-commutation implies that there is no non-trivial “time-reversal” operation in the primitive algebra that restores both carrier and ledger to their original values.

Formally:

**Theorem 3.3.6 (Absence of non-trivial time-reversing words).**  
Let (\mathcal{C}\_0) be an admissible carrier and (\Pi \in \mathcal{O}^*) an admissible word such that the resulting carrier (\mathcal{C}\_m = \Pi(\mathcal{C}\_0)) satisfies  
[  
I(\mathcal{C}\_m) > I(\mathcal{C}\_0).  
]  
Then there is no word (\Pi^{-1} \in \mathcal{O}^*) built from the same primitive set with the following properties:

1. (\Pi^{-1}(\mathcal{C}\_m)) is admissible and equal to (\mathcal{C}\_0),
2. Capacity is conserved at every step,
3. Ledger rules of Section 3.2 are obeyed at every step.

In other words, once record has strictly increased, there is no admissible sequence of primitive operations that returns both the carrier and the ledger to their original state.

*Sketch of proof.*

Assume such a (\Pi^{-1}) exists. Concatenate to form (\Pi^{-1}\Pi), which is an admissible word taking (\mathcal{C}\_0) back to itself. By capacity conservation and the ledger rules, record cannot decrease, so along (\Pi^{-1}\Pi) we must have (I(\mathcal{C}\_0) \le I(\mathcal{C}\_m)) and (I) non-decreasing at every step. But we also require (I(\mathcal{C}\_0)) at the end, so strict increase on (\Pi) cannot be undone by non-decreasing steps—contradiction. (\square)

This theorem expresses in algebraic terms that **record growth is fundamentally irreversible** under the primitive dynamics as defined.

**3.3.6 Arrow of time as induced partial order**

The monotonicity of (I) suggests a natural **partial order** on carriers reachable from a given initial state.

**Definition 3.3.7 (Ledger order).**  
Fix an initial carrier (\mathcal{C}\_0). For any two carriers (\mathcal{C},\mathcal{C}' \in \mathcal{R}(\mathcal{C}\_0)), define  
[  
\mathcal{C} \preceq \mathcal{C}'  
]  
if there exists an admissible path from (\mathcal{C}) to (\mathcal{C}') with capacity conservation at each step and  
[  
I(\mathcal{C}') \ge I(\mathcal{C}).  
]

**Proposition 3.3.8 (Ledger order is a partial order).**  
On the reachability class (\mathcal{R}(\mathcal{C}\_0)), the relation (\preceq) is reflexive, transitive, and antisymmetric (up to neutral moves), so it defines a partial order modulo neutral path equivalence.

*Sketch.*

* Reflexive: empty word gives (\mathcal{C} \preceq \mathcal{C}).
* Transitive: concatenation of paths with non-decreasing (I) yields a path with non-decreasing (I).
* Antisymmetric up to neutral moves: if (I(\mathcal{C}') \ge I(\mathcal{C})) and (I(\mathcal{C}) \ge I(\mathcal{C}')), then (I(\mathcal{C}) = I(\mathcal{C}')); combined with the absence of non-trivial time-reversal, paths in both directions must differ only by neutral moves. (\square)

This partial order is the algebraic expression of the **intrinsic arrow of time**: carriers with larger (I) are “later” in the sense of the irreversible growth of record.

**3.3.7 Summary**

Section 3.3 has established:

* Non-commutation of renewal and sink operations in relevant regimes.
* Idempotent or semi-idempotent behaviors for synchronization and strong framing.
* Global monotonicity of the record ledger along admissible paths.
* The impossibility of undoing genuine record growth with the same primitive operator set.
* A natural partial order on carriers induced by record growth, which encodes an intrinsic arrow of time.

In the next section (3.4), we will introduce **interval components** (\Delta t(\nu)), (\Delta \tau(\nu)), and (|\Delta x(\nu)|) as functions of flip-count vectors, and show how a Lorentz-like invariant interval arises from this operator algebra and the ledger structure.

**3.4 Invariant Interval from Flip Counts**

We now introduce the **interval components** (\Delta t), (\Delta \tau), and (|\Delta x|) as functions of flip-count vectors, and state the existence of a Lorentz-like **invariant quadratic form**. This construction depends only on the flip-counts and the ledger/arrow structure developed in Sections 3.1–3.3.

**3.4.1 Interval components as functions of (\nu)**

Recall that for any operator word (\Pi \in \mathcal{O}^\*) we defined its flip-count vector  
[  
\nu(\Pi) = (\nu\_F,\nu\_S,\nu\_T,\nu\_C,\nu\_{CT}) \in \mathbb{N}^5.  
]

We now define three real-valued functions of (\nu):

**Definition 3.4.1 (Interval components).**  
There exist functions  
[  
\Delta t,;\Delta \tau,;|\Delta x| : \mathbb{N}^5 \to \mathbb{R}\_{\ge 0}  
]  
such that for any word (\Pi) we write  
[  
\Delta t(\Pi) := \Delta t(\nu(\Pi)),\quad  
\Delta \tau(\Pi) := \Delta \tau(\nu(\Pi)),\quad  
|\Delta x(\Pi)| := |\Delta x|(\nu(\Pi)).  
]

Interpretation:

* (\Delta t(\Pi)) is the **coordinate-time** increment associated with the flip sequence (\Pi).
* (\Delta \tau(\Pi)) is the **proper-time** increment.
* (|\Delta x(\Pi)|) is the **spatial separation magnitude**.

At this stage these are abstract labels: we do not assume any particular analytic form; we only require consistency conditions stated below.

**3.4.2 Additivity and neutrality**

The interval components are required to be additive under concatenation of words and insensitive to neutral moves.

Let (\Pi\_1,\Pi\_2) be operator words, and let (\Pi := \Pi\_2\Pi\_1) be their concatenation.

**Axiom 3.4.2 (Additivity in flip-count space).**  
For all (\nu\_1,\nu\_2 \in \mathbb{N}^5),  
[  
\Delta t(\nu\_1 + \nu\_2) = \Delta t(\nu\_1) + \Delta t(\nu\_2),  
]  
[  
\Delta \tau(\nu\_1 + \nu\_2) = \Delta \tau(\nu\_1) + \Delta \tau(\nu\_2),  
]  
[  
|\Delta x|(\nu\_1 + \nu\_2) \le |\Delta x|(\nu\_1) + |\Delta x|(\nu\_2),  
]  
with equality in the last relation for collinear compositions (see below).

Equivalently, for words (\Pi\_1,\Pi\_2),  
[  
\Delta t(\Pi\_2\Pi\_1) = \Delta t(\Pi\_2) + \Delta t(\Pi\_1),  
]  
and similarly for (\Delta \tau), with (|\Delta x|) subadditive.

**Axiom 3.4.3 (Neutral invariance).**  
If (\Pi\_0) is neutral on the reachability class of an initial carrier (in the sense of Definition 2.4.5), then for any (\Pi) such that (\Pi\_0\Pi) is defined and admissible,  
[  
\Delta t(\Pi\_0\Pi) = \Delta t(\Pi),\quad  
\Delta \tau(\Pi\_0\Pi) = \Delta \tau(\Pi),\quad  
|\Delta x|(\Pi\_0\Pi) = |\Delta x|(\Pi).  
]

Thus, the interval components depend only on the **flip-count class** of a word and not on neutral rearrangements.

**3.4.3 Collinearity and triangle-like inequality**

The subadditivity of (|\Delta x|) reflects a triangle-like inequality in the space of flip-counts.

**Definition 3.4.4 (Collinear compositions).**  
Two words (\Pi\_1,\Pi\_2) are said to be **collinear** (relative to an initial carrier) if the spatial separation associated with their concatenation is exactly the sum of their individual separations:  
[  
|\Delta x(\Pi\_2\Pi\_1)| = |\Delta x(\Pi\_2)| + |\Delta x(\Pi\_1)|.  
]

In that case, we can regard the associated spatial displacements as “pointing in the same direction.” In the absence of collinearity, the inequality  
[  
|\Delta x(\Pi\_2\Pi\_1)| < |\Delta x(\Pi\_2)| + |\Delta x(\Pi\_1)|  
]  
captures non-collinear composition.

We do not construct a full vector-valued (\Delta \mathbf{x}) here; it is enough to know that the magnitude (|\Delta x|) has a well-defined norm-like behavior with respect to concatenation.

**3.4.4 Existence of a quadratic invariant**

We now state the existence of a **quadratic invariant** built from (\Delta t), (\Delta \tau), and (|\Delta x|). This is the key structural statement: the operator algebra induces a Minkowski-like metric relation on the interval components.

**Axiom 3.4.5 (Quadratic invariant).**  
There exists a positive constant (c > 0) such that for all flip-count vectors (\nu),  
[  
\Delta t(\nu)^2 = \Delta \tau(\nu)^2 + c^{-2}|\Delta x(\nu)|^2.  
]

Equivalently, for any word (\Pi),  
[  
\Delta t(\Pi)^2 = \Delta \tau(\Pi)^2 + c^{-2}|\Delta x(\Pi)|^2.  
]

This is the **invariant interval relation**. The constant (c) is interpreted as the characteristic conversion factor between spatial and temporal units in the emergent kinematics.

We emphasize:

* The existence of such a relation is taken as a structural axiom of the V1 formalism.
* No numeric value is assigned to (c) in this volume; any calibration is reserved for empirical work.

**3.4.5 Path-independence within flip-count classes**

We also require that the interval is **path-independent** within a given flip-count class.

Let (\mathcal{C}*0) be an initial admissible carrier, and let (\mathcal{R}*\nu(\mathcal{C}\_0)) be the set of carriers reachable with flip-count (\nu).

**Proposition 3.4.6 (Path-independence on (\mathcal{R}\_\nu)).**  
For any two admissible paths (\Pi\_1,\Pi\_2) out of (\mathcal{C}\_0) with  
[  
\nu(\Pi\_1) = \nu(\Pi\_2) = \nu,  
]  
we have  
[  
\Delta t(\Pi\_1) = \Delta t(\Pi\_2),\quad  
\Delta \tau(\Pi\_1) = \Delta \tau(\Pi\_2),\quad  
|\Delta x(\Pi\_1)| = |\Delta x(\Pi\_2)|.  
]

*Sketch.*

By construction, (\Delta t,\Delta \tau,|\Delta x|) are functions of (\nu) alone. Therefore, if two words share the same flip-count vector, they yield identical values of these components. Neutral words do not change the components. (\square)

Thus, the interval components are well-defined on **flip-count classes**, and the invariant relation of Axiom 3.4.5 holds for all carriers in (\mathcal{R}\_\nu(\mathcal{C}\_0)).

**3.4.6 Lorentz-like symmetry group in interval space**

The quadratic invariant defines a natural symmetry group acting on the interval components. Consider the set of triples ((\Delta t,\Delta \tau,|\Delta x|)) that can arise from admissible flip-counts. The invariant relation  
[  
\Delta t^2 - c^{-2}|\Delta x|^2 = \Delta \tau^2  
]  
is preserved under a family of linear transformations mixing (\Delta t) and spatial components (once a full vector (\Delta \mathbf{x}) is introduced) that form a Lorentz-like group.

We do not need the explicit representation in this section; it is enough to note that:

* Any transformation between frames (CSs) that respects the operator algebra and ledger structure must **preserve the invariant interval**.
* This defines a symmetry group analogous to the usual Lorentz group acting on spacetime intervals, but here it is derived from the flip-count structure and ledger axioms.

These frame transformations will be associated more concretely with Collective Spheres in Section 3.5.

**3.4.7 Summary**

Section 3.4 has:

* Attached interval components (\Delta t(\nu), \Delta \tau(\nu), |\Delta x(\nu)|) to flip-count vectors (\nu).
* Required additivity and neutral invariance of these components.
* Stated the existence of a quadratic invariant  
  [  
  \Delta t^2 = \Delta \tau^2 + c^{-2}|\Delta x|^2  
  ]  
  as a structural axiom of the theory.
* Established path-independence within flip-count classes.
* Noted the existence of a Lorentz-like symmetry group acting on these interval components.

In the next subsection (3.5), we will connect this invariant interval to **Collective Spheres (CS)** and define frames and frame transformations in terms of CS structure, thereby completing the algebraic layer that underpins the emergent kinematics.

**3.5 Frames and Frame Transformations via Collective Spheres**

Sections 3.1–3.4 defined the operator algebra, ledger, and invariant interval purely in terms of flip-counts and carriers. We now connect these structures to **Collective Spheres (CS)** and make precise what is meant by a **frame** and a **frame transformation** in the V1 formalism.

The central idea is:

Frames are CS-based choices of coordinates on the space of interval components that preserve the invariant quadratic form defined in Section 3.4.

**3.5.1 Frames as CS-based coordinate systems**

Recall from Section 1.4 that a Collective Sphere (\mathrm{CS}) at tick (k) is a synchronized collection of PMSs sharing a compatible boundary description. For the algebraic purposes of this section, we suppress the tick index and treat (\mathrm{CS}) as the **carrier** of a family of admissible paths.

**Definition 3.5.1 (Frame associated with a CS).**  
A **frame** (\mathcal{F}) associated with a Collective Sphere (\mathrm{CS}) consists of:

1. A choice of an initial carrier (\mathcal{C}*0 \in \mathcal{C}*{\mathrm{adm}}) representing a reference PMS within (\mathrm{CS}).
2. A map  
   [  
   \Phi\_{\mathcal{F}} : {\Pi \in \mathcal{O}^\* \mid \Pi(\mathcal{C}*0) \text{ admissible}} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}*{\ge 0}  
   ]  
   defined by  
   [  
   \Phi\_{\mathcal{F}}(\Pi) := \bigl(\Delta t(\Pi), \Delta \tau(\Pi), |\Delta x(\Pi)|\bigr),  
   ]  
   where the interval components are those defined in Section 3.4.
3. A convention for interpreting the triple ((\Delta t,\Delta \tau,|\Delta x|)) as **coordinates** ((t,\tau,x)) within that frame.

We call (\mathcal{F}) a **CS-frame** if:

* The admissible words (\Pi) considered by (\Phi\_{\mathcal{F}}) correspond to paths whose carriers remain within the reachability class associated with (\mathrm{CS}), and
* The invariant relation  
  [  
  \Delta t^2 = \Delta \tau^2 + c^{-2}|\Delta x|^2  
  ]  
  holds for all such paths.

Intuitively, a frame is a way of *coordinatizing* the flip-count-based intervals available to a particular synchronized cluster of PMSs.

**3.5.2 Frame coordinates and origin choice**

Within a CS-frame (\mathcal{F}), we often interpret:

* (\Delta t(\Pi)) as the **frame time coordinate** difference between the initial carrier (\mathcal{C}\_0) and (\Pi(\mathcal{C}\_0)).
* (|\Delta x(\Pi)|) as the **spatial separation magnitude** in that frame.

We can then define a coordinate map  
[  
(t,x,\tau) : \mathcal{R}(\mathcal{C}*0) \to \mathbb{R} \times \mathbb{R}*{\ge 0} \times \mathbb{R}\_{\ge 0}  
]  
by choosing a fixed origin (\mathcal{C}\_0) and setting, for any reachable carrier (\mathcal{C} = \Pi(\mathcal{C}\_0)),  
[  
t(\mathcal{C}) := \Delta t(\Pi), \quad  
\tau(\mathcal{C}) := \Delta \tau(\Pi), \quad  
x(\mathcal{C}) := |\Delta x(\Pi)|.  
]

By path-independence within flip-count classes (Proposition 3.4.6), these definitions do not depend on the particular admissible word (\Pi) used to reach (\mathcal{C}), only on the associated flip-count vector (\nu(\Pi)).

We thus obtain a **coordinate chart** on the reachability class of (\mathcal{C}\_0) within a given CS, in which each carrier is assigned coordinates ((t,x,\tau)) obeying  
[  
t^2 = \tau^2 + c^{-2}x^2.  
]

**3.5.3 Frame transformations as invariance-preserving maps**

Given two frames (\mathcal{F}) and (\mathcal{F}') (typically associated with different CSs, or with different choices of reference carrier within the same CS), a **frame transformation** is a map between their coordinate descriptions that preserves the invariant interval.

**Definition 3.5.2 (Frame transformation).**  
Let (\mathcal{F},\mathcal{F}') be two frames with coordinate assignments  
[  
\Phi\_{\mathcal{F}}(\Pi) = (\Delta t,\Delta \tau,|\Delta x|),  
]  
[  
\Phi\_{\mathcal{F}'}(\Pi) = (\Delta t',\Delta \tau',|\Delta x'|)  
]  
for admissible paths (\Pi) in a common domain.

A **frame transformation** (L : \mathcal{F} \to \mathcal{F}') is a map such that for all admissible (\Pi),  
[  
(\Delta t',\Delta \tau',|\Delta x'|) = L(\Delta t,\Delta \tau,|\Delta x|)  
]  
and the invariant interval is preserved:  
[  
\Delta t'^2 = \Delta \tau'^2 + c^{-2}|\Delta x'|^2  
\quad \Leftrightarrow \quad  
\Delta t^2 = \Delta \tau^2 + c^{-2}|\Delta x|^2.  
]

We denote the set of all such transformations (on a given domain of interval components) by  
[  
\mathcal{G}\_{\mathrm{CS}},  
]  
and call it the **frame symmetry group** associated with the corresponding CS context.

We do not need an explicit matrix representation of (\mathcal{G}\_{\mathrm{CS}}) in this volume; it suffices to know:

* (\mathcal{G}\_{\mathrm{CS}}) consists precisely of those maps that preserve the quadratic invariant;
* Under suitable assumptions on spatial degrees of freedom, (\mathcal{G}\_{\mathrm{CS}}) can be identified with a Lorentz-like group acting on a vector space that refines ((\Delta t, \Delta \mathbf{x})).

**3.5.4 Consistency with the ledger and arrow**

Frame transformations must be compatible with the **ledger-induced arrow of time**. That is, if (\mathcal{C}) and (\mathcal{C}') lie in the reachability class of an initial carrier and are related by the ledger order (\mathcal{C} \preceq \mathcal{C}') (Definition 3.3.7), then any acceptable frame transformation should respect that ordering in the time coordinates.

Formally:

**Axiom 3.5.3 (Arrow compatibility).**  
Let (\mathcal{C},\mathcal{C}') be carriers in the reachability class of (\mathcal{C}\_0) with (\mathcal{C} \preceq \mathcal{C}'). For any pair of frames (\mathcal{F}, \mathcal{F}') and any frame transformation (L : \mathcal{F} \to \mathcal{F}'),

* if (t(\mathcal{C}) \le t(\mathcal{C}')) in frame (\mathcal{F}), then
* (t'(\mathcal{C}) \le t'(\mathcal{C}')) in frame (\mathcal{F}'),

where (t, t') are the time coordinates in (\mathcal{F}) and (\mathcal{F}'), respectively.

In other words, frame transformations preserve the **direction** of the ledger order; they do not invert the arrow of time. This is consistent with Theorem 3.3.6: no time-reversing transformation exists within the primitive algebra that restores both carriers and ledger values.

**3.5.5 Frames within a single CS vs between CSs**

Within a single CS, frame differences typically arise from:

* different choices of reference carrier (\mathcal{C}\_0), or
* different ways of splitting interval components into time and space coordinates (e.g., rotations or boosts in an emergent spacetime sense).

Between different CSs, frame transformations can also encode:

* relative motion (in the emergent kinematic sense),
* differences in synchronization conventions, or
* differing coarse-grainings of the same underlying operator paths.

In both cases, the key requirement is the same:

* frame transformations are maps within (\mathcal{G}\_{\mathrm{CS}}) that preserve the invariant interval and the ledger-induced arrow.

**3.5.6 Summary**

Section 3.5 has:

* Defined **frames** as CS-based coordinate systems that attach interval components ((\Delta t,\Delta \tau,|\Delta x|)) to operator paths.
* Introduced **frame transformations** as maps between such coordinate systems that preserve the invariant quadratic form.
* Identified the associated **symmetry group** (\mathcal{G}\_{\mathrm{CS}}) as the set of all invariance-preserving transformations on the interval space.
* Imposed compatibility between these transformations and the ledger-induced arrow of time.

With this, the **algebraic kinematics** of the V1 formalism is complete: the primitive operators, ledger, and flip-count-based interval together generate an intrinsic time arrow and a Lorentz-like structure of frames and frame changes, all constructed without assuming any pre-existing spacetime background.

**4. Fractal Inner Geometry & Pivot Gate**

**4.1 IN as IFS & Dimension (D)**

Up to this point, (\mathrm{IN}\_k) has been treated as an abstract “record” container with no internal structure specified. In this section we endow (\mathrm{IN}\_k) with a **fractal geometric structure** by modeling it as the attractor of an **iterated-function system** (IFS). This lets us assign an effective **fractal dimension** (D) to (\mathrm{IN}\_k), which will later serve as the key argument of the pivot function (g(D)).

**4.1.1 Inner network as an IFS attractor**

We begin by associating to each (\mathrm{IN}\_k) an underlying metric space and a contractive IFS.

**Definition 4.1.1 (Underlying metric space for IN).**  
For each admissible carrier (\mathcal{C}\_k = (k,h\_k,\mathrm{IN}\_k,\mathrm{ON}\_k)), there exists a metric space ((X\_k,d\_k)) such that:

* The elements of (\mathrm{IN}\_k) can be represented as points (or structured subsets) of (X\_k).
* The metric (d\_k) captures a notion of “closeness” of record elements at tick (k).

We do not fix the nature of (X\_k) (it may be a subset of (\mathbb{R}^n), a manifold, or a more abstract metric space); all that is required is that (X\_k) supports an iterated-function system with a well-defined attractor.

**Definition 4.1.2 (IN as IFS attractor).**  
An **iterated-function system (IFS)** on ((X\_k,d\_k)) is a finite collection of contractions  
[  
\mathcal{F}*k = { f*{k,1}, f\_{k,2}, \dots, f\_{k,M\_k} },  
]  
each (f\_{k,i} : X\_k \to X\_k) satisfying  
[  
d\_k(f\_{k,i}(x), f\_{k,i}(y)) \le s\_{k,i} , d\_k(x,y)  
]  
for some contraction factor (0 \le s\_{k,i} < 1).

The **attractor** (K\_{\mathrm{IN},k} \subseteq X\_k) of (\mathcal{F}*k) is the unique non-empty compact set satisfying  
[  
K*{\mathrm{IN},k} = \bigcup\_{i=1}^{M\_k} f\_{k,i}(K\_{\mathrm{IN},k}).  
]

We assume:

* The inner network (\mathrm{IN}\_k) is representable (up to isomorphism) as such an attractor:  
  [  
  \mathrm{IN}*k \cong K*{\mathrm{IN},k}.  
  ]

Thus, the “shape” of the record at tick (k) is modeled as a fractal set generated by repeated application of contractions on a metric space.

**4.1.2 Effective fractal dimension (D)**

We now define the effective fractal dimension of (\mathrm{IN}\_k), denoted (D\_k) or simply (D) when context is clear.

**Definition 4.1.3 (Effective dimension of IN).**  
The **effective fractal dimension** (D\_k) of the inner network at tick (k) is defined as the (appropriate) fractal dimension of the attractor (K\_{\mathrm{IN},k}). Depending on the regularity of the IFS and the underlying space, this may be:

* the Hausdorff dimension,
* or, in simpler self-similar cases, the solution (D) of the Moran equation  
  [  
  \sum\_{i=1}^{M\_k} s\_{k,i}^D = 1.  
  ]

In this volume we do not commit to a specific fractal-dimension definition in all generality; we only require that:

1. For each (\mathrm{IN}*k), there is a well-defined scalar (D\_k \in [D*{\min},D\_{\max}]) measuring its effective fractal “thickness”.
2. The range of interest is bounded:  
   [  
   1 \le D\_{\min} \le D\_k \le D\_{\max} \le 3,  
   ]  
   reflecting the fact that IN structures relevant to the model lie between line-like and volume-filling regimes.

When no ambiguity arises, we drop the tick index and simply write (D) for the effective dimension associated with the IN under consideration.

**4.1.3 Dimension as a function on carriers**

To make the dependence on carriers explicit, we treat the dimension as a function  
[  
D : \mathcal{C}*{\mathrm{adm}} \to [D*{\min}, D\_{\max}],  
]  
with  
[  
D(\mathcal{C}\_k) = D\_k.  
]

We assume:

* (D) depends only on the IN part of the carrier (and possibly coarse context information), i.e.  
  [  
  D(\mathcal{C}\_k) = D(\mathrm{IN}\_k),  
  ]  
  and is invariant under neutral moves that leave (\mathrm{IN}\_k) unchanged (up to isomorphism).
* Along an admissible operator path (\mathcal{C}\_0 \to \mathcal{C}*1 \to \dots), the dimension may change as a function of the operators applied (for example, if the structure of (\mathrm{IN}k) is refined or coarse-grained), but it remains within the fixed interval ([D{\min},D*{\max}]).

Later, when we introduce the discrete context ladder and its dimension curve (D(n)), we will consider a **coarse-grained dimension** at each band (n), which can be viewed as an average or effective value of (D) over carriers in that band.

**4.1.4 Compatibility with operator actions**

The primitive operators act on carriers and thus may change the inner network and its dimension. We impose broad compatibility assumptions without fixing detailed rules for each operator here.

**Axiom 4.1.4 (Dimension compatibility with primitives).**  
Let (\mathcal{C}\_k) be admissible and let (O(\mathcal{C}*k) = \mathcal{C}*{k'}) be defined for some primitive (O \in {F,S,T,C,CT}). Then:

1. The effective dimension changes continuously or in small discrete steps:  
   [  
   |D(\mathcal{C}*{k'}) - D(\mathcal{C}k)| \le \Delta D{\max},  
   ]  
   for some fixed bound (\Delta D*{\max} > 0). This encodes the idea that individual tick updates do not radically change the fractal nature of IN.
2. Neutral moves (words that act as identity on the carrier class) do not change (D):  
   [  
   D(\Pi\_0(\mathcal{C}\_k)) = D(\mathcal{C}\_k)  
   ]  
   for any neutral (\Pi\_0) on the relevant reachability class.
3. For operators that merely reorganize or synchronize without fundamentally altering the internal fractal structure (e.g., idealized (T) and (C) in neutral regimes), we may treat  
   [  
   D(\mathcal{C}\_{k'}) = D(\mathcal{C}\_k)  
   ]  
   to a good approximation.

Thus, (D) behaves as a slowly varying or bandwise constant quantity under the basic dynamics.

**4.1.5 Dimension range and hinge precondition**

We have not yet singled out (D=2) as special—that will happen when we introduce the pivot function (g(D)) and collapse kernels. However, we can already state a **hinge precondition** on the allowed range of (D).

**Axiom 4.1.5 (Hinge precondition on dimension).**

1. The interval ([D\_{\min},D\_{\max}]) includes the value 2:  
   [  
   D\_{\min} < 2 < D\_{\max}.  
   ]
2. There exists at least one context (and often a distinguished one) in which the effective dimension of IN satisfies  
   [  
   D = 2.  
   ]

This axiom captures, at the structural level, the idea that **a 2-dimensional regime of the inner network is available** and will later serve as a pivot or hinge when we introduce context ladders and collapse kernels.

**4.1.6 Summary**

Section 4.1 has:

* Modeled the inner network (\mathrm{IN}\_k) as the attractor of an iterated-function system on a metric space ((X\_k,d\_k)).
* Assigned to (\mathrm{IN}*k) an effective fractal dimension (D\_k \in [D*{\min},D\_{\max}] \subseteq [1,3]).
* Treated the dimension as a function (D(\mathcal{C})) on admissible carriers that is stable under neutral moves and varies only modestly under primitive operations.
* Imposed a hinge precondition that the value (D=2) lies within the allowed range and is realized in at least one context.

In the next subsection (4.2), we will introduce the **pivot function** (g(D)), which takes the fractal dimension as input and outputs a coupling weight that will control the strength of interactions and context couplings throughout the rest of the theory.

**4.2 Pivot Function (g(D))**

Having assigned an effective fractal dimension (D) to the inner network (\mathrm{IN}), we now introduce the **pivot function** (g(D)). This is a scalar weight that encodes how strongly a given IN-configuration participates in interactions and couplings. It will appear throughout the theory:

* as a **scale factor** in actions and Lagrangians,
* as a **gate weight** in context couplings,
* and as a **pivot marker** at the hinge value (D=2).

In this section we define (g(D)), state its structural properties, and indicate its intended uses, without fixing any specific analytic form.

**4.2.1 Basic definition and domain**

We treat the effective dimension (D) as lying in a fixed interval  
[  
D \in [D\_{\min}, D\_{\max}] \subseteq [1,3],  
]  
with (2 \in (D\_{\min},D\_{\max})) and at least one context realizing (D=2) (Axiom 4.1.5).

**Definition 4.2.1 (Pivot function).**  
The **pivot function**  
[  
g : [D\_{\min}, D\_{\max}] \to \mathbb{R}\_{>0}  
]  
assigns to each admissible value of the inner-network dimension (D) a positive scalar weight (g(D)), subject to the following constraints:

1. **Positivity**  
   [  
   g(D) > 0 \quad \text{for all } D \in [D\_{\min}, D\_{\max}].  
   ]
2. **Normalization at the hinge**  
   [  
   g(2) = 1.  
   ]
3. **Continuity**  
   [  
   g \text{ is continuous on } [D\_{\min}, D\_{\max}].  
   ]
4. **Local regularity near (D=2)**  
   There exists an (\varepsilon > 0) such that (g) is differentiable on ((2-\varepsilon,2+\varepsilon)).

No explicit closed form for (g) is assumed in this volume; only these structural properties are required.

**4.2.2 Symmetry and hinge-centered behavior**

We regard (D=2) as a **pivot** or **hinge** value of the inner dimension. The function (g(D)) is intended to measure “distance from the pivot” in an effective way. To capture this, we place mild restrictions on how (g) behaves as (D) moves away from 2.

**Axiom 4.2.2 (Hinge-centered behavior).**

1. **Minimum at the pivot**  
   (g) has a local extremum at (D=2), and we choose the convention that this is a **reference value** (g(2)=1). Whether this is a strict minimum or simply a distinguished normalization depends on later usage; structurally we require that (D=2) is a special point.
2. **Monotonicity in (|D-2|)**  
   There exists some radius (\varepsilon>0) such that for (D) in ((2-\varepsilon,2+\varepsilon)),  
   [  
   |D\_1 - 2| < |D\_2 - 2| ;\Rightarrow; |g(D\_1) - 1| \le |g(D\_2) - 1|.  
   ]  
   That is, departures of (g(D)) from the normalized value 1 grow (weakly) with (|D-2|) close to the hinge.
3. **Symmetry (optional but natural)**  
   In many constructions it is convenient (though not strictly required) to assume a hinge-centered symmetry  
   [  
   g(2+\delta) \approx g(2-\delta)  
   ]  
   for small (|\delta|). When this is assumed, it reflects a structural reciprocity between “thicker-than-2D” and “thinner-than-2D” inner networks.

These conditions ensure that (g(D)) behaves smoothly near the hinge and that the hinge plays a qualitatively distinguished role.

**4.2.3 Interpretation as a coupling weight**

Intuitively, the pivot function translates fractal “thickness” into an **interaction weight**:

* When (D \approx 2), the inner network is at the hinge dimension, and  
  [  
  g(D) \approx 1,  
  ]  
  so couplings are at their reference strength.
* When (D) deviates from 2, the weight (g(D)) may increase or decrease, indicating:
  + **enhanced coupling** in some regimes (e.g. more record “surface” available for interaction), or
  + **suppressed coupling** in others (e.g. more volume-like or filamentary structure reducing effective contact).

We purposely do **not** fix whether (g(D)) grows or shrinks with (|D-2|) in general; different sectors of the theory may use different conventions (e.g. (g>1) for certain bands, (g<1) for others), as long as the structural constraints above are satisfied.

**4.2.4 Use in context couplings**

One major role of (g(D)) is in controlling **cross-context couplings**—interactions between carriers whose inner networks have different effective dimensions.

The basic pattern is:

* Within a single **Collective Sphere (CS)** at a given context band, the relevant dimension (D\_{\mathrm{CS}}) sets an internal weight:  
  [  
  g\_{\mathrm{CS}} := g(D\_{\mathrm{CS}}).  
  ]
* For interactions **between** two contexts (A) and (B), with effective dimensions (D\_A,D\_B), we use a composite weight of the form  
  [  
  g\_{A \leftrightarrow B} = \varphi\bigl(g(D\_A), g(D\_B)\bigr),  
  ]  
  where (\varphi) is a symmetric, positive function (e.g. (\min), harmonic mean, or product) that respects the normalization at the hinge.

In later sections, we will specialize to simple choices, such as  
[  
g\_{A \leftrightarrow B} = \min(g(D\_A), g(D\_B)),  
]  
for concreteness. For now, we note only that **all such composite weights are built from (g(D))** and inherit its structural properties (positivity, normalization at the hinge, smooth behavior near (D=2)).

**4.2.5 Use in actions and Lagrangians**

The second primary role of (g(D)) is as a **scale factor in actions and Lagrangians**:

* In the discrete context ladder (later in Section 6–7), each band (n) has an associated dimension (D(n)). The contribution from that band to the master action is weighted by (g(D(n))).
* In continuum embeddings, a Lagrangian density (\mathcal{L}) is often multiplied by a factor (g(D)) or (g(D(n))) to encode the **effective strength** of fields or interactions at that dimension.

Schematically, terms of the form  
[  
S\_{\text{band}} \sim g(D(n)) \cdot \text{(band-specific action contribution)}  
]  
appear throughout the theory. The pivot normalization (g(2)=1) then enforces that the **hinge band** uses the unscaled Lagrangian, while other bands are weighted relative to that reference.

**4.2.6 Compatibility with collapse and kernels (preview)**

Later, when we construct:

* **Collapse kernels** on the PMS boundary (Section 11.2), and
* **Null-cone kernels** and retarded operators (Section 11.5),

the dimension (D) enters those kernels, and (g(D)) acts as a **modulating factor**. At the hinge (D=2), the kernel reduces to a constant projector (identity on appropriate modes), and the weight (g(2)=1) marks this as the natural reference case.

We do not give kernel formulas in this subsection; we only record that:

* collapse behavior depends smoothly on (D), and
* (g(D)) is the scalar “dial” controlling how strongly different dimensions contribute in the action-level and operator-level constructions.

**4.2.7 Summary**

Section 4.2 has:

* Defined the pivot function (g(D)) as a positive, continuous weight on the dimension interval ([D\_{\min},D\_{\max}]), with normalization (g(2)=1).
* Imposed hinge-centered behavior: departures from the pivot grow with (|D-2|) near the hinge, and a symmetric behavior (g(2+\delta) \approx g(2-\delta)) is natural (though not strictly required).
* Interpreted (g(D)) as a **coupling weight** used in:
  + cross-context interactions, and
  + band-weighted action and Lagrangian terms.
* Noted its compatibility with later collapse-kernel and null-cone constructions, where the hinge (D=2) and the normalization (g(2)=1) play a central role.

In the next subsection (4.3), we will refine the description of time structure by introducing **dual exponents** for record and potential, and define an effective **present dimension** that will typically sit near (D=2), further motivating the hinge interpretation.

**4.3 Dual Time Exponents & Present Dimension**

The fractal dimension (D) of the Inner Network describes how record structure scales in space. To capture how **record** and **potential** scale in “time-like” directions (past and future), we introduce a pair of **dual exponents** and combine them into an effective **present-moment exponent**. This present exponent will be aligned with the hinge value (D=2).

**4.3.1 Dual exponents (d\_{\text{past}}) and (d\_{\text{future}})**

The Inner Network (IN) and Outer Network (ON) do not grow in the same way as ticks progress:

* IN (record) tends to **thicken and accumulate** structure.
* ON (potential) tends to be **sparser and branching**, representing possibilities rather than fixed structure.

We encode this asymmetry with two exponents.

**Definition 4.3.1 (Dual time exponents).**

We introduce two effective scaling exponents:

* (d\_{\text{past}} > 1): characterizes the growth/density of record.
* (d\_{\text{future}} < 1): characterizes the spread/sparsity of potential.

They are not identical to the fractal dimension (D) of IN; rather, they are **effective exponents** that describe how much structure is available in:

* the **past direction** (record, IN), and
* the **future direction** (potential, ON).

Informally:

* (d\_{\text{past}} > 1) means the record side **thickens faster than linearly** under scaling (more structure per “unit” than a simple 1D chain).
* (d\_{\text{future}} < 1) means the potential side remains **thin or sparse** compared to a full volume (less structure per unit, reflecting branching possibilities rather than filled regions).

We do **not** assign specific numerical values here; only the inequalities and the qualitative ordering  
[  
d\_{\text{past}} > 1 > d\_{\text{future}}  
]  
are part of the core structure.

**4.3.2 Present-moment exponent (d\_{\text{PMS}})**

The present moment is the interface where past record and future potential meet. Its effective scaling combines the two exponents above into a single **present-moment exponent**.

**Definition 4.3.2 (Present-moment exponent).**

We define  
[  
d\_{\text{PMS}} := d\_{\text{past}} + d\_{\text{future}} - 1.  
]

The subtraction by 1 reflects that the present is **not** just the sum of two independent directions; it is a **boundary** where record and potential share degrees of freedom. In other words, the “overlap” between past and future directions reduces the naïve sum by one effective dimension.

Structurally, the theory assumes that:

1. The present-moment exponent sits near a **pivot value**,
2. This pivot is identified with an effective dimension close to 2:  
   [  
   d\_{\text{PMS}} \approx 2.  
   ]

We then align:

* the present-moment exponent (d\_{\text{PMS}}), and
* the hinge value of the IN fractal dimension (D),

by identifying the hinge band with  
[  
D\_{\text{hinge}} = 2 \quad\text{and}\quad d\_{\text{PMS}} \approx 2.  
]

Thus, the present can be treated as a **two-dimensional effective interface** between:

* an inward, thickening record side (governed by (d\_{\text{past}}>1)), and
* an outward, branching potential side (governed by (d\_{\text{future}}<1)).

**4.3.3 Role and interpretation**

These dual exponents play several roles in the overall structure:

1. **Encoding the arrow of time**  
   The inequality  
   [  
   d\_{\text{past}} > d\_{\text{future}}  
   ]  
   says that record accumulates “faster” or more thickly than potential spreads. This mirrors the ledger asymmetry:
   * more and more structure is stored in IN,
   * while ON is continually sampled, committed, and partially discarded.

The dual exponents therefore restate the **time arrow** in geometric terms: the past is fractally “heavier” than the future.

1. **Hinge behavior at (D \approx 2)**  
   The near-equality  
   [  
   d\_{\text{PMS}} \approx 2  
   ]  
   identifies a pivot layer where:
   * inward folding (record accumulation) and
   * outward branching (potential spread)

are in a balanced tension. This is precisely the hinge that appears in:

* + the dimension profile (D(r) = 2 + \delta(r)), and
  + the pivot function (g(D)) with normalization (g(2)=1).

Contexts whose IN dimension (D) sits near this present exponent are treated as **maximally coupled** (pivot band), while those with dimensions far from 2 are more attenuated.

1. **Bridge between dynamics and geometry**  
   The dual exponents translate:
   * **dynamical asymmetry** (irreversible ON → IN flow, record growth),

into

* + **geometric asymmetry** (different effective scaling of record vs potential).

This link allows the present-moment interface to be treated simultaneously as:

* + a **geometric pivot** (hinge in the dimension profile and gate function), and
  + a **dynamical pivot** (where flips, ledger changes, and framing events are focused).

In summary, (d\_{\text{past}}, d\_{\text{future}}, d\_{\text{PMS}}) formalize the idea that:

* the past is a thick, folded, high-density record,
* the future is a thin, branching, low-density potential, and
* the present sits at a hinge layer whose effective exponent aligns with the pivotal dimension (D \approx 2) used throughout the fractal and gate structure of the theory.

In the next subsection (4.4), we will introduce a minimal **complex structure** on an effective “present plane,” which provides the smallest amount of additional structure needed to talk about complex amplitudes and Born-style weights while remaining consistent with this dual-exponent picture.

**4.4 Present Plane & Complex Structure (J)**

The dual exponents (d\_{\text{past}}) and (d\_{\text{future}}) capture an asymmetry between record (IN) and potential (ON). To talk about **superposition-like structure** and **Born-style weights** at the hinge, we introduce a minimal **present plane** equipped with a **complex structure** (J). This is the smallest amount of extra structure needed to support complex amplitudes without assuming a full Hilbert space from the outset.

**4.4.1 The present plane as a 2D real space**

We begin by abstracting a 2-dimensional real vector space that represents “present-moment directions” at the hinge.

**Definition 4.4.1 (Present plane).**  
Let (\mathcal{P}) be a 2-dimensional real vector space. Elements of (\mathcal{P}) are called **present vectors**. We use:

* Real coordinate pairs ((x,y) \in \mathbb{R}^2), or
* Abstract notation (v \in \mathcal{P}),

interchangeably, once a basis is chosen.

The intuition is:

* One “direction” encodes a real-valued **weight** or **density** related to record;
* The orthogonal direction encodes a conjugate aspect (e.g. phase-like information) needed for interference-like behavior.

At this stage we do **not** interpret (\mathcal{P}) as a spatial plane; it is an internal present-plane attached to the hinge layer.

**4.4.2 Complex structure (J) on (\mathcal{P})**

We now endow (\mathcal{P}) with a complex structure—an operator that squares to (-\mathrm{id}).

**Definition 4.4.2 (Complex structure).**  
A **complex structure** on (\mathcal{P}) is a linear map  
[  
J : \mathcal{P} \to \mathcal{P}  
]  
such that:

1. (J^2 = -\mathrm{id}\_{\mathcal{P}}),
2. (J) is invertible, with (J^{-1} = -J).

Given a choice of basis ({e\_1,e\_2}), we can represent (J) as multiplication by the matrix  
[  
J \sim \begin{pmatrix}  
0 & -1 \  
1 & 0  
\end{pmatrix},  
]  
but no particular basis is privileged in the formalism.

The pair ((\mathcal{P},J)) is then a real 2D vector space that can be identified with (\mathbb{C}) by treating the action of (J) as multiplication by (i). Concretely, given any (v \in \mathcal{P}), we can write  
[  
v = x e\_1 + y e\_2  
\quad \longleftrightarrow \quad  
z = x + i y \in \mathbb{C},  
]  
and (J(v)) corresponds to (i z).

**4.4.3 Present amplitudes as vectors in (\mathcal{P})**

To connect the present plane to the AR structure, we associate **present amplitudes** with certain context-resolved outcomes or “branches.”

Let ({R\_i}) be a finite or countable family of disjoint regions (or basins) on the IN attractor for a given context, representing a coarse partition of possible record outcomes at the hinge.

**Definition 4.4.3 (Present amplitudes).**  
For each region (R\_i), we assign a vector (v\_i \in \mathcal{P}), called a **present amplitude**, with the properties:

1. Each (v\_i) is non-zero whenever (R\_i) is admissible in the current ON/IN configuration.
2. The collection ({v\_i}) is normalized in some context-dependent way (e.g. via a constraint on the sum of squared norms; details appear in Section 4.6).

Via an identification (\mathcal{P} \cong \mathbb{C}), we can regard these as complex amplitudes:  
[  
v\_i ;\longleftrightarrow; a\_i \in \mathbb{C},  
]  
but the theory only needs the real 2D structure and the action of (J).

The purpose of these amplitudes is to:

* encode **interference-capable structure** in the present layer, and
* provide the basis for **Born-style weightings** that will be tied to the measure of IN basins in Section 4.6.

**4.4.4 Compatibility with flip algebra and ladder structure**

The present plane is not an independent degree of freedom; it must be compatible with:

* the **flip algebra** (operator words and flip counts), and
* the **context ladder** and dimension profile.

We encode this by requiring that:

1. **Flip-induced transformations** act linearly on (\mathcal{P}):  
   For each primitive operator (O) and each context in which present amplitudes are defined, there exists a linear map  
   [  
   L\_O : \mathcal{P} \to \mathcal{P}  
   ]  
   such that acting with (O) on carriers corresponds (at the hinge) to acting with (L\_O) on present amplitudes.
2. **Context shifts** (moving up/down the ladder) map present amplitudes to present amplitudes via linear operators that respect the complex structure:  
   [  
   L\_{\text{context}} J = J L\_{\text{context}}.  
   ]  
   That is, the context shifts are (\mathbb{C})-linear with respect to (J).
3. **Neutral moves** act trivially on present amplitudes:  
   If (\Pi\_0) is neutral on the relevant reachability class, then the induced transformation on (\mathcal{P}) is the identity:  
   [  
   L\_{\Pi\_0} = \mathrm{id}\_{\mathcal{P}}.  
   ]

These conditions guarantee that the complex structure is **consistent** with the operator algebra and does not introduce extra path-dependence beyond what is already encoded by flip counts.

**4.4.5 Not yet a full Hilbert space**

Although ((\mathcal{P},J)) and the present amplitudes ({v\_i}) can be identified with a complex 1D or low-dimensional vector space, we deliberately do **not** posit a full Hilbert-space structure here:

* No global inner product on (\mathcal{H}) is assumed.
* We only require:
  + a 2D real vector space for each relevant present context,
  + a complex structure (J),
  + and linear maps induced by primitive operators and context shifts.

A full Hilbert space will emerge effectively in later sections when:

* amplitudes are organized into larger spaces indexed by outcome sets ({R\_i}),
* a norm or inner product is introduced to express Born-style probabilities, and
* the context ladder provides a natural decomposition of such spaces.

For now, the complex structure (J) is the **minimal piece** that allows us to:

* treat present amplitudes as complex-like objects,
* perform interference-like linear combinations, and
* square their norms in a way that can be tied to IN measures.

**4.4.6 Summary**

Section 4.4 has:

* Introduced the **present plane** (\mathcal{P}) as a 2D real vector space.
* Equipped (\mathcal{P}) with a complex structure (J) satisfying (J^2=-\mathrm{id}), making (\mathcal{P}) effectively a copy of (\mathbb{C}).
* Associated **present amplitudes** (v\_i \in \mathcal{P}) with outcome basins (R\_i) on the IN attractor at the hinge.
* Required that primitive operators and context shifts act linearly on (\mathcal{P}) and commute with (J), while neutral moves act as the identity.
* Emphasized that this is **not yet** a full Hilbert space; it is a minimal complex structure sufficient for later Born-style weighting rules.

In the next subsection (4.5), we will define the **context gate rule**, which uses the pivot function (g(D)) to weight couplings within and between contexts, and show how this gate interacts with the present-plane amplitudes.

**4.5 Context Gate Rule**

We now define the **context gate rule**, which uses the pivot function (g(D)) to weight interactions:

* **within** a single context (or CS) whose IN dimension is (D), and
* **between** two contexts with dimensions (D\_A) and (D\_B).

The gate rule provides a systematic way to attach **effective coupling strengths** to transitions and amplitudes, based solely on the inner-network dimensions and the hinge normalization (g(2)=1).

**4.5.1 Single-context gate**

Let (\mathcal{C}) be an admissible carrier whose inner network has effective dimension  
[  
D(\mathcal{C}) = D.  
]  
This carrier belongs to some context (e.g. a band on the ladder, or a CS at that band). The **single-context gate weight** is simply  
[  
g\_{\text{ctx}}(\mathcal{C}) := g(D).  
]

This number is used as:

* a **multiplicative factor** on amplitudes or interaction terms associated with that context, and
* a **band weight** in actions and Lagrangians when summing over context contributions.

At the hinge, (D=2), we have (g\_{\text{ctx}} = 1), so the hinge band is unscaled; other bands are weighted relative to this reference.

**4.5.2 Cross-context gate between two contexts**

Consider two carriers (\mathcal{C}\_A,\mathcal{C}\_B) in (possibly different) contexts (A,B), with effective dimensions  
[  
D\_A := D(\mathcal{C}\_A),\quad D\_B := D(\mathcal{C}\_B).  
]

We define a **cross-context gate weight** (g\_{A\leftrightarrow B}) capturing how strongly these contexts can interact. We require this weight to satisfy:

1. **Positivity**  
   [  
   g\_{A\leftrightarrow B} > 0.  
   ]
2. **Symmetry**  
   [  
   g\_{A\leftrightarrow B} = g\_{B\leftrightarrow A}.  
   ]
3. **Normalization at the hinge**  
   If (D\_A = D\_B = 2),  
   [  
   g\_{A\leftrightarrow B} = 1.  
   ]
4. **Monotone weakening away from the hinge**  
   For contexts with dimensions farther from the hinge, the effective gate is further from the reference value 1.

A simple and structurally convenient choice is:

**Definition 4.5.1 (Cross-context gate weight).**  
For contexts (A,B) with dimensions (D\_A,D\_B), define  
[  
g\_{A\leftrightarrow B} := \min\bigl(g(D\_A), g(D\_B)\bigr).  
]

This choice satisfies:

* symmetry and positivity,
* normalization (g\_{A\leftrightarrow B} = 1) when both (D\_A=D\_B=2),
* and the intuitive property that the **less strongly coupled** context (smaller (g)) limits the joint coupling.

Other symmetric combinations (e.g. product, harmonic mean) could be used, but (\min) captures the structural intent with minimal assumptions. The core requirement is that all such combinations are built from (g(D)) and inherit its hinge-centered behavior.

**4.5.3 Gate action on amplitudes in the present plane**

Let (\mathcal{P}) be the present plane with complex structure (J) (Section 4.4), and let (v \in \mathcal{P}) be a present amplitude associated with an outcome basin in a given context.

**Single-context gate.**  
For a carrier (\mathcal{C}) with dimension (D), the gate acts on amplitudes by scalar multiplication:  
[  
v ;\mapsto; G\_{\text{ctx}}(\mathcal{C}), v := g(D), v.  
]  
This simply rescales the amplitude by the context’s gate weight.

**Cross-context gate.**  
For a pair of contexts (A,B) with amplitudes (v\_A,v\_B \in \mathcal{P}) and weight  
[  
g\_{A\leftrightarrow B} = \min(g(D\_A),g(D\_B)),  
]  
a simple gate action on a combined amplitude (v\_{AB} \in \mathcal{P}) is  
[  
v\_{AB} ;\mapsto; G\_{A\leftrightarrow B}, v\_{AB} := g\_{A\leftrightarrow B}, v\_{AB}.  
]

In both cases:

* The gate acts **linearly** on (\mathcal{P}),
* Commutes with the complex structure:  
  [  
  G,J = J,G,  
  ]  
  since it is scalar multiplication, and
* Is invariant under neutral moves at the carrier level, provided those neutral moves do not change (D).

**4.5.4 Gate-modulated operator action**

We can incorporate the gate weights into the effective action of primitive operators. For a primitive operator (O) acting in a context (A) with dimension (D\_A):

* At the carrier level, we continue to write  
  [  
  \mathcal{C}' = O(\mathcal{C}).  
  ]
* At the amplitude level on (\mathcal{P}), we write the induced linear map as  
  [  
  v ;\mapsto; L\_O^{(A)} v,  
  ]  
  and define the **gated operator**  
  [  
  \widetilde{L}\_O^{(A)} := g(D\_A), L\_O^{(A)}.  
  ]

For an interaction between contexts (A) and (B), the gated operator on a combined amplitude would be  
[  
\widetilde{L}*O^{(A\leftrightarrow B)} := g*{A\leftrightarrow B}, L\_O^{(A\leftrightarrow B)}.  
]

Thus, (g(D)) does not modify the **form** of the operator; it **scales** its effective strength according to the inner dimension of the contexts involved.

**4.5.5 Neutrality at the hinge**

At the hinge dimension (D=2), the gate becomes neutral:

* For a single context with (D=2),  
  [  
  g(D) = g(2) = 1 \quad \Rightarrow \quad G\_{\text{ctx}} = \mathrm{id}\_{\mathcal{P}}.  
  ]
* For cross-context interactions with both dimensions equal to 2,  
  [  
  g\_{A\leftrightarrow B} = 1 \quad \Rightarrow \quad G\_{A\leftrightarrow B} = \mathrm{id}\_{\mathcal{P}}.  
  ]

This reflects the idea that the hinge band is the **reference layer**: interactions there are not scaled up or down. All other context bands are weighted relative to this pivot.

**4.5.6 Summary**

Section 4.5 has:

* Defined **single-context** gate weights (g\_{\text{ctx}}(\mathcal{C}) = g(D(\mathcal{C}))).
* Defined **cross-context** gate weights (g\_{A\leftrightarrow B}) as symmetric, positive combinations of (g(D\_A)) and (g(D\_B)), with a concrete choice (g\_{A\leftrightarrow B} = \min(g(D\_A),g(D\_B))).
* Specified how gate weights act **linearly** on present-plane amplitudes and commute with the complex structure (J).
* Shown how primitive operator actions on amplitudes can be **modulated** by gate weights, without changing their structural form.
* Highlighted that the hinge (D=2) is **gate-neutral**, with (g(2)=1).

In the next subsection (4.6), we will combine the present-plane structure, outcome basins on IN, and the gate rule to formulate a **Born-style weighting** that ties squared amplitude norms to the measure of IN basins, purely as a structural feature of the AR formalism.

**4.6 Born-Style Weighting (Structural)**

With the present plane (\mathcal{P}), complex structure (J), and gate rule in place, we can now state the **Born-style weighting** that ties:

* **present amplitudes** (vectors in (\mathcal{P}))  
  to
* **record basins** on the IN attractor,

purely as a structural rule of the theory. This gives the AR analogue of the Born rule without appealing to any specific experiment or dataset.

**4.6.1 Outcome basins on IN and a reference measure**

Fix a context (e.g. a band on the ladder with IN attractor (K\_{\mathrm{IN}})) and partition its inner network into disjoint **outcome basins**:  
[  
K\_{\mathrm{IN}} = \bigsqcup\_{i \in I} R\_i,  
]  
where each (R\_i) is a measurable region representing a coarse-grained “outcome” of interest in that context.

**Definition 4.6.1 (Reference measure on IN).**  
We assume that there is a Borel probability measure  
[  
\mu\_{\mathrm{IN}} : \mathcal{B}(K\_{\mathrm{IN}}) \to [0,1]  
]  
defined on the (\sigma)-algebra of Borel subsets (\mathcal{B}(K\_{\mathrm{IN}})), such that:

* (\mu\_{\mathrm{IN}}(K\_{\mathrm{IN}}) = 1),
* (\mu\_{\mathrm{IN}}) is invariant under the IFS dynamics that generates (K\_{\mathrm{IN}}) (i.e. it is an invariant or stationary measure for the inner IFS).

For each outcome basin (R\_i), the measure (\mu\_{\mathrm{IN}}(R\_i)) is interpreted as the **geometric weight** of that basin within the IN attractor.

**4.6.2 Amplitudes attached to basins**

As in Section 4.4, we attach present-plane amplitudes to these basins.

**Definition 4.6.2 (Amplitude assignment).**  
To each basin (R\_i) we associate a vector (v\_i \in \mathcal{P}). We require:

1. (v\_i \neq 0) whenever (R\_i) is admissible (non-empty and dynamically allowed) in the current context.
2. The family ({v\_i}*{i\in I}) is* ***normalized*** *with respect to a fixed norm (|\cdot|) on (\mathcal{P}) (induced by any Euclidean inner product compatible with (J)), so that  
   [  
   \sum*{i \in I} |v\_i|^2 = 1.  
   ]

Given the identification (\mathcal{P} \cong \mathbb{C}), we can equivalently think of (v\_i) as a complex amplitude (a\_i), with (|v\_i|^2 = |a\_i|^2); but the theory only needs the norm and the complex structure (J), not the full Hilbert-space machinery at this stage.

**4.6.3 Structural Born-style rule**

The **Born-style rule** is the structural prescription that relates amplitude norms to outcome weights.

**Axiom 4.6.3 (Born-style weighting).**  
For a fixed context with outcome basins (R\_i), reference measure (\mu\_{\mathrm{IN}}), and amplitudes (v\_i \in \mathcal{P}) satisfying (\sum\_i |v\_i|^2 = 1), the probability weight (p\_i) assigned to outcome (R\_i) is  
[  
p\_i := |v\_i|^2.  
]

Moreover, in **equilibrium** configurations of the context (where the amplitude assignment is stable under repeated application of the relevant framing and sink operations), these weights satisfy  
[  
p\_i = \mu\_{\mathrm{IN}}(R\_i).  
]

In other words:

* **Formally:** probabilities are given by squared norms of present-plane amplitudes.
* **Geometrically:** in stable contexts, these probabilities match the normalized measure of the associated basins on the IN attractor.

This is the AR analogue of the Born rule:

* The amplitude structure (through (|v\_i|^2)) and the fractal geometry of IN (through (\mu\_{\mathrm{IN}}(R\_i))) are **tied together** by the requirement that they agree in equilibrium.

**4.6.4 Neutral invariance and gate neutrality (single context)**

Within a single context (fixed (D)), the gate weight (g(D)) is a common scalar factor and therefore cancels in normalized probabilities:

* If all amplitudes are multiplied by the same positive scalar (\lambda), normalization rescales them back so that  
  [  
  p\_i = \frac{|\lambda v\_i|^2}{\sum\_j |\lambda v\_j|^2} = \frac{|v\_i|^2}{\sum\_j |v\_j|^2}.  
  ]

Thus:

* **Neutral moves** on carriers that do not change IN or amplitudes leave the ({p\_i}) unchanged.
* **Single-context gating** (using (g(D))) leaves the normalized probabilities unchanged, since it rescales all amplitudes in that context by the same factor.

The gate becomes important when comparing amplitudes **between** different contexts or bands (Section 4.5), not when normalizing probabilities **within** a single context at fixed (D).

**4.6.5 Coarse-graining and additivity over basins**

A useful consistency requirement is that probabilities respect coarse-graining of basins.

Let (\alpha \subset I) be an index subset, and define a coarse-grained basin  
[  
R\_\alpha := \bigsqcup\_{i \in \alpha} R\_i.  
]

We can define a coarse-grained amplitude (v\_\alpha) and corresponding probability (p\_\alpha).

**Definition 4.6.4 (Coarse-grained amplitude and probability).**  
Given a collection ({R\_i,v\_i}*{i\in I}), define  
[  
v*\alpha := \sum\_{i \in \alpha} v\_i,  
]  
and  
[  
p\_\alpha := |v\_\alpha|^2.  
]

We require that in decohered or effectively orthogonal contexts (where cross-terms between different (v\_i) vanish or average to zero under framing):

* Probabilities for disjoint basins **add**:  
  [  
  p\_\alpha = \sum\_{i \in \alpha} p\_i,  
  ]  
  and  
  [  
  \mu\_{\mathrm{IN}}(R\_\alpha) = \sum\_{i \in \alpha} \mu\_{\mathrm{IN}}(R\_i).  
  ]

Thus, the Born-style rule is consistent with coarse-graining of outcomes and with the additive structure of the IN measure.

**4.6.6 Structural role in the theory**

Summarizing the role of the Born-style weighting:

1. It **links** the minimal complex structure on the present plane ((\mathcal{P},J)) to the fractal geometry of IN via a reference measure (\mu\_{\mathrm{IN}}).
2. It expresses the AR version of the Born rule:
   * probabilities are squared norms of amplitudes,
   * in equilibrium they match geometric weights of outcome basins.
3. It is **compatible** with:
   * neutral moves (no change to ({p\_i})),
   * single-context gating (overall factors cancel),
   * and coarse-graining of outcomes (additivity over disjoint unions).

No empirical examples are needed here; the Born-style rule is treated as an internal structural relation between amplitudes and IN geometry. Empirical tests of this relation, and concrete realizations in specific systems, belong to companion volumes focused on evidence and applications.

With this, the **inner fractal geometry + present-plane + gate** layer is complete. In the next major part (Section 5), we will move from the continuous radial picture (D(r)) and dual exponents to the **discrete context ladder**, define (D(n)) and the reproduction kernel, and build the master action on the ladder that underlies the field-theoretic and gravitational structures later in the theory.

**5. Unified Fractal–Sink Paradigm & Radial Profile (D(r))**

**5.1 Dual Exponents Across a Context Ladder**

In Sections 3 and 4, we introduced two complementary ingredients:

* the **sink/renew algebra** and ledger, which encode an intrinsic arrow of time via monotone growth of record (I) and corresponding reduction of exposure (E);
* the **fractal geometry of IN**, with an effective inner dimension (D) and dual time exponents (d\_{\text{past}}>1), (d\_{\text{future}}<1), combined into a present exponent (d\_{\text{PMS}}\approx 2).

We now unify these into a **radial picture**: a single parameter (r) that labels context “distance” from the present hinge, along which:

* the **sink/renew balance** varies,
* the **dual exponents** become functions of (r), and
* the **inner dimension profile** (D(r)) will later be defined.

This section introduces the dual exponents as functions of (r); later subsections will specify (D(r)), the boundary projector (\mathcal{B}), and the double-flip + projection theorem at the hinge.

**5.1.1 Radial parameter (r) as context distance**

We begin by formalizing the radial parameter that will underlie the profile (D(r)).

**Definition 5.1.1 (Radial parameter).**  
Let (r \in \mathbb{R}) (or (\mathbb{Z}) in a discrete version) be a **radial context parameter** with:

* (r=0) representing the **present hinge layer**,
* (r<0) representing **inward** (finer) contexts,
* (r>0) representing **outward** (coarser) contexts.

We interpret (r) as an abstract coordinate on a one-dimensional manifold of contexts. It is related to, but not identical with, the ladder index (n):

* later, we will assume a monotone map (r \leftrightarrow n) so that “more inward” and “more outward” directions match between the continuous and discrete descriptions.

At this stage, (r) is **dimensionless** and purely structural; no specific physical units or scales are attached.

**5.1.2 Radial families of carriers and IN/ON balance**

For each value of (r), we consider an equivalence class of carriers that “sit” at that radial context.

**Definition 5.1.2 (Radial carrier class).**  
Fix an initial admissible carrier (\mathcal{C}\_0) at (r=0). For each (r), let (\mathcal{R}^{(r)}(\mathcal{C}\_0)) denote a class of carriers reachable from (\mathcal{C}\_0) via admissible operations such that:

* the carriers in (\mathcal{R}^{(r)}) share similar coarse-grained IN/ON structure appropriate to radial level (r),
* transitions between different (\mathcal{R}^{(r)}) classes (as (r) varies) correspond to **coarse-graining** or **refinement** operations (not just tick evolution).

Intuitively:

* (\mathcal{R}^{(0)}) collects carriers at the present hinge;
* (\mathcal{R}^{(r<0)}) collects inward refinements (finer “inside” structure);
* (\mathcal{R}^{(r>0)}) collects outward aggregations (coarser “enclosing” structure).

We do not specify an explicit equivalence relation here; only that such classes exist and they allow us to treat record/potential growth exponents and dimensions as **functions of (r)**.

**5.1.3 Dual exponents as functions of (r)**

We now promote the dual time exponents introduced in Section 4.3 to radial functions.

**Definition 5.1.3 (Radial dual exponents).**  
For each radial context (r), we define:

* (d\_{\text{past}}(r) > 1): an effective exponent characterizing how **record** (IN) accumulated up to radial level (r) scales with coarse-graining.
* (d\_{\text{future}}(r) < 1): an effective exponent characterizing how **potential** (ON) available beyond level (r) scales with coarse-graining.

These exponents summarize, at each (r), the relative “thickness” of the record and the “thinness” of the potential, from the vantage point of the present hinge.

We assume:

* **Continuity / slow variation:**  
  (d\_{\text{past}}(r)) and (d\_{\text{future}}(r)) vary continuously (or in small discrete steps) with (r). Abrupt large discontinuities are excluded except at specially marked transitions, which we do not introduce in this volume.
* **Asymptotic behavior (qualitative):**
  + For sufficiently inward contexts (r \ll 0):  
    record has been heavily folded and accumulated, so (d\_{\text{past}}(r)) tends toward a “dense” regime (approaching a volume-like dimension), while potential in those inner bands is limited, so (d\_{\text{future}}(r)) may decrease further.
  + For sufficiently outward contexts (r \gg 0):  
    record is coarser and more aggregate, so (d\_{\text{past}}(r)) may drop toward lower effective dimensions, while potential spread is large but thin, keeping (d\_{\text{future}}(r)) below 1.

We do not fix concrete functional forms; only this qualitative monotone behavior relative to inward/outward direction and the inequalities (d\_{\text{past}}(r) > 1 > d\_{\text{future}}(r)).

**5.1.4 Present exponent as a radial combination**

At each radial level, we define a **present-moment exponent** (d\_{\text{PMS}}(r)) by combining the dual exponents.

**Definition 5.1.4 (Radial present exponent).**  
For each (r), define  
[  
d\_{\text{PMS}}(r) := d\_{\text{past}}(r) + d\_{\text{future}}(r) - 1.  
]

This mirrors Definition 4.3.2, but now as a function of radial context. The subtraction by 1 again reflects the idea that the present interface is not the simple sum of independent past/future directions; some degrees of freedom are shared across the boundary.

We impose:

**Axiom 5.1.5 (Hinge condition on (d\_{\text{PMS}}(r))).**

1. At the hinge (r=0),  
   [  
   d\_{\text{PMS}}(0) = 2.  
   ]
2. For small (|r|),  
   [  
   d\_{\text{PMS}}(r) = 2 + \epsilon(r),  
   ]  
   where (\epsilon(r)) is a small correction function with (\epsilon(0)=0) and (|\epsilon(r)|) bounded in a neighborhood of 0.

Thus, near the hinge, the present interface has an effective exponent very close to 2, matching the hinge dimension that will later be assigned to the IN fractal profile (D(r)).

**5.1.5 Relation to sink/renew balance along (r)**

The radial dual exponents are meant to capture how much **sink** vs **renew** action has been effectively accumulated inward of a given radial level.

Qualitatively:

* For a fixed present hinge, moving inward ((r \to r-|\delta r|)):
  + the **cumulative impact of sink** operations grows, as more ON has been committed to IN in the inward direction;
  + the effective exponent (d\_{\text{past}}(r)) tends to increase (record more folded/dense), while (d\_{\text{future}}(r)) may reduce (less new potential remains inside that layer).
* Moving outward ((r \to r+|\delta r|)):
  + the direct impact of sink diminishes; the record appears more coarse and aggregated;
  + (d\_{\text{past}}(r)) may decline toward values closer to 1 (less dense than a volume), while potential spread (d\_{\text{future}}(r)) remains below 1 but may vary more slowly.

This suggests an indirect relation between the ledger and dual exponents:

* larger **cumulative record** in inward bands ↔ higher (d\_{\text{past}}(r)),
* smaller residual potential ↔ lower (d\_{\text{future}}(r)).

We do not write this as an exact formula; rather, we treat (d\_{\text{past}}(r)) and (d\_{\text{future}}(r)) as **effective summaries** of the long-run action of the flip algebra along radial contexts.

**5.1.6 Alignment with the IN dimension profile (D(r))**

In Section 4.1 we introduced the effective fractal dimension (D) for IN; here we prepare to treat it as a function of radial context (D(r)).

The dual exponents and the IN dimension profile are aligned by a simple matching condition at the hinge:

**Axiom 5.1.6 (Present–IN matching at the hinge).**

At (r=0),  
[  
D(0) = d\_{\text{PMS}}(0) = 2.  
]

Away from the hinge, the theory treats (D(r)) and (d\_{\text{PMS}}(r)) as related but not necessarily identical:

* they may differ by correction terms associated with how IN geometry is projected onto the present interface,
* but they share the same qualitative behavior (e.g. both approach more volume-like or more filamentary regimes as (r) moves inward or outward).

Formally, we can write  
[  
D(r) = d\_{\text{PMS}}(r) + \delta\_D(r),  
]  
where (\delta\_D(r)) is a small correction function with (\delta\_D(0)=0). In many regimes we may treat (\delta\_D(r)) as negligible and simply approximate (D(r) \approx d\_{\text{PMS}}(r)).

**5.1.7 Summary**

Section 5.1 has:

* Introduced a **radial context parameter** (r) that organizes contexts inward and outward from the present hinge.
* Defined **radial carrier classes** (\mathcal{R}^{(r)}) that associate carriers to each radial context.
* Promoted the dual time exponents to radial functions (d\_{\text{past}}(r) > 1), (d\_{\text{future}}(r) < 1), describing record vs potential “thickness” at each (r).
* Defined a **radial present exponent** (d\_{\text{PMS}}(r) = d\_{\text{past}}(r) + d\_{\text{future}}(r) - 1) with the hinge condition (d\_{\text{PMS}}(0) = 2).
* Connected these exponents qualitatively to the cumulative action of sink and renew along radial contexts.
* Aligned the present exponent with the IN dimension profile at the hinge via (D(0) = d\_{\text{PMS}}(0) = 2).

In the next subsection (5.2), we will make the **dimension profile** explicit by introducing (D(r) = 2 + \delta(r)), describing its qualitative shape across radial contexts, and setting up the conditions that will later support the discrete ladder curve (D(n)) and the hinge projector.

**5.2 Dimension Profile (D(r) = 2 + \delta(r))**

We now introduce an explicit **dimension profile** (D(r)) for the inner network across radial contexts. This profile encodes how the effective fractal dimension of IN changes as we move inward or outward from the present hinge at (r=0).

**5.2.1 Definition of the dimension profile**

We model the IN dimension as a function of the radial parameter (r):

**Definition 5.2.1 (Dimension profile).**  
Let  
[  
D : \mathbb{R} \to [D\_{\min}, D\_{\max}] \subseteq [1,3]  
]  
be a continuous function assigning an effective IN dimension (D(r)) to each radial context (r). We write  
[  
D(r) = 2 + \delta(r),  
]  
where (\delta(r)) is a real-valued **deviation function** satisfying:

1. **Hinge condition**  
   [  
   D(0) = 2 \quad\Longleftrightarrow\quad \delta(0) = 0.  
   ]
2. **Bounded deviations**  
   [  
   D\_{\min} \le D(r) \le D\_{\max}  
   ]  
   for all (r), with (1 \le D\_{\min} < 2 < D\_{\max} \le 3).

The function (\delta(r)) measures how far the IN dimension at radial level (r) is from the hinge value (2).

**5.2.2 Qualitative shape: interior volume vs exterior filament**

The dimension profile is meant to reflect a qualitative picture:

* **Inward** ((r \ll 0)): IN is more volume-filling or “dense,” so (D(r)) tends toward a higher value (approaching 3).
* **Outward** ((r \gg 0)): IN is more filamentary or sparse in the outward direction, so (D(r)) tends toward a lower value (approaching 1).
* **At the hinge** ((r=0)): IN is effectively 2-dimensional, so (D(0)=2).

We encode this qualitatively as:

**Axiom 5.2.2 (Asymptotic behavior).**

1. There exist limits (possibly approached asymptotically)  
   [  
   \lim\_{r\to -\infty} D(r) = D\_{\text{in}} \in (2,3], \quad  
   \lim\_{r\to +\infty} D(r) = D\_{\text{out}} \in [1,2),  
   ]  
   with (D\_{\text{in}} > 2 > D\_{\text{out}}).
2. The profile is **monotone in the large**:
   * For sufficiently negative (r), (D(r)) is non-increasing as (r) increases toward 0 (coming in from the interior).
   * For sufficiently positive (r), (D(r)) is non-increasing as (r) increases further outward (falling off toward filamentary regimes).

We do not require strict monotonicity at every point; local plateaus and small fluctuations are allowed as long as the overall trend is respected.

**5.2.3 Smoothness near the hinge**

Near the hinge, we want (D(r)) to be smooth enough that we can:

* treat small shifts in (r) as small changes in the inner dimension, and
* meaningfully expand (D(r)) in a Taylor series around (r=0).

**Axiom 5.2.3 (Local smoothness around (r=0)).**  
There exists an (\varepsilon > 0) such that:

1. (D(r)) is differentiable on ((- \varepsilon, \varepsilon)),
2. (\delta(r) = D(r) - 2) admits a Taylor expansion  
   [  
   \delta(r) = \delta'(0), r + \tfrac{1}{2}\delta''(0), r^2 + O(r^3),  
   ]  
   with (\delta(0) = 0).

In many idealized constructions, one may further impose (\delta'(0)=0) (to make the hinge a local extremum), but the core formalism does not strictly require this; it only requires that (r=0) is a distinguished point where (D=2).

**5.2.4 Logistic-type and mirror-symmetric profiles (optional structure)**

For some later constructions (e.g. in the discrete ladder (D(n))), it is convenient to use a **logistic-like** or **mirror-symmetric** profile. This is not strictly necessary for the V1 theory, but it provides a useful template.

A canonical example is:  
[  
D(r) = 1 + \frac{2}{1 + e^{k(r-r\_0)}},  
]  
with (r\_0 = 0), which yields:

* (D(0)=2),
* (D(r)\to 3) as (r\to -\infty),
* (D(r)\to 1) as (r\to +\infty).

In this case, (\delta(r) = D(r)-2) is negative for (r>0) and positive for (r<0), and the profile is **mirror-symmetric** around (r=0) in the sense that  
[  
D(-r) + D(r) = 4,  
]  
or equivalently  
[  
\delta(-r) = -\delta(r).  
]

While the exact logistic form and antisymmetry are not imposed as axioms, the theory accommodates such choices and often uses them as concrete realizations in examples.

**5.2.5 Alignment with dual exponents and present exponent**

As introduced in Section 5.1, each radial context has dual exponents (d\_{\text{past}}(r)) and (d\_{\text{future}}(r)), and a present exponent  
[  
d\_{\text{PMS}}(r) = d\_{\text{past}}(r) + d\_{\text{future}}(r) - 1.  
]

We align the dimension profile with the present exponent via:

**Axiom 5.2.4 (Dimension–present alignment).**

There exists a correction function (\delta\_D(r)) with (\delta\_D(0)=0) such that  
[  
D(r) = d\_{\text{PMS}}(r) + \delta\_D(r)  
]  
for all (r), with (|\delta\_D(r)|) bounded and small on the radial range of interest.

In the simplest approximation,  
[  
D(r) \approx d\_{\text{PMS}}(r),  
]  
and at the hinge this reduces to the explicit match  
[  
D(0) = d\_{\text{PMS}}(0) = 2.  
]

Thus, the **same hinge** appears in both:

* the fractal profile of the inner network, and
* the combined dual-exponent description of record/potential scaling.

**5.2.6 Correspondence to ladder bands**

The continuous profile (D(r)) provides the template for the **discrete ladder profile** (D(n)) introduced later:

* Each band index (n) is associated with a radial value (r(n)), and we define  
  [  
  D(n) := D\bigl(r(n)\bigr).  
  ]
* The hinge band (n=0) is mapped to (r(0)=0), so  
  [  
  D(0) = D(r(0)) = 2.  
  ]
* Inward bands (n<0) correspond to (r<0) (higher dimensions, volume-like regimes); outward bands (n>0) correspond to (r>0) (lower dimensions, filamentary regimes).

In Section 6, we will specify a logistic-type shape for (D(n)) on the discrete ladder, which is the bandwise analogue of the radial profile described here.

**5.2.7 Summary**

Section 5.2 has:

* Defined a radial **dimension profile** (D(r) = 2 + \delta(r)) on the inner network, with hinge value (D(0)=2).
* Specified qualitative asymptotic behavior:
  + (D(r) \to D\_{\text{in}}>2) inward,
  + (D(r) \to D\_{\text{out}}<2) outward.
* Imposed **local smoothness** near the hinge, allowing Taylor expansion around (r=0).
* Described optional logistic/mirror-symmetric realizations that capture the same qualitative behavior.
* Aligned (D(r)) with the radial present exponent (d\_{\text{PMS}}(r)), especially at the hinge.
* Prepared the way for the discrete ladder profile (D(n)) via a mapping (r \leftrightarrow n).

In the next subsection (5.3), we will introduce the **boundary projector** (\mathcal{B}) that collapses radial structure onto the hinge layer and state the conditions under which applying (\mathcal{B}) after double flips (e.g. (F\circ F) or (S\circ S)) yields matched distributions at the pivot, laying the groundwork for the double-flip + projection theorem.

**5.3 Boundary Projector (\mathcal{B}) and Pivot Meaning**

The radial profile (D(r)) describes how the inner-network dimension changes as we move inward or outward from the present hinge. To connect this continuous structure back to the **hinge layer** where many of the core constructions (present plane, gate, Born-style rule) live, we introduce a **boundary projector** (\mathcal{B}).

Intuitively, (\mathcal{B}) “forgets” radial detail and collapses a radially extended configuration down to its **hinge boundary** at (r=0). This section defines (\mathcal{B}), states its properties, and explains how it contributes to the special role of the pivot.

**5.3.1 Radial extension and hinge layer**

We first formalize the idea that structures (carriers, IN configurations) can be **radially extended**, and the hinge layer is a distinguished slice of that extension.

Let:

* (\mathcal{R}^{(r)}(\mathcal{C}\_0)) be the radial carrier class at context parameter (r) (Definition 5.1.2), and
* (D(r)) the dimension profile from Section 5.2.

A “radially extended” description of a configuration can be thought of as a **collection** ({\mathcal{C}^{(r)}}\_r) of carriers indexed by (r), tied together by compatibility relations (coarse-graining inward, refinement outward, etc.). We will not model this collection in full detail; for our purposes, it is enough to know that:

* for each (r), there is a representative carrier (\mathcal{C}^{(r)}), and
* the hinge layer corresponds to the representative at (r=0), denoted (\mathcal{C}^{(0)}).

The boundary projector (\mathcal{B}) is the operator that, given such extended data (or any object that carries radial dependence), returns the **hinge-layer projection**.

**5.3.2 Abstract definition of (\mathcal{B})**

We treat (\mathcal{B}) as an operator that acts on any object with radial dependence (e.g. radial carriers, IN configurations, or functions of (r)) and “evaluates” it at the hinge layer in a suitably coarse-grained way.

**Definition 5.3.1 (Boundary projector).**  
Let (\mathcal{X}) be a space of radially extended objects of the form  
[  
X = { X(r) \mid r \in \mathbb{R} },  
]  
where each (X(r)) is a context-level structure (e.g. a carrier, an IN configuration, or a suitable functional thereof). The **boundary projector**  
[  
\mathcal{B} : \mathcal{X} \to \mathcal{X}\_0  
]  
is a map into a “hinge space” (\mathcal{X}\_0) such that:

1. **Hinge evaluation (formal part)**  
   For any (X \in \mathcal{X}),  
   [  
   \mathcal{B}[X] = X(0)  
   ]  
   when (X(r)) is already parameterized as a simple function of (r).
2. **Coarse-grained hinge projection (general part)**  
   In more general cases where (X) is not a simple pointwise function of (r) (e.g. if it contains self-consistency conditions or averaging across radial neighborhoods), (\mathcal{B}[X]) is defined as the **coarse-grained hinge representative** that encodes the effective structure at (r=0) after integrating out radial detail.

In both cases:

* (\mathcal{B}) is **idempotent**:  
  [  
  \mathcal{B}(\mathcal{B}[X]) = \mathcal{B}[X].  
  ]
* (\mathcal{B}) acts as an identity on objects that already live purely at the hinge (no radial dependence):  
  [  
  X \in \mathcal{X}\_0 ;\Rightarrow; \mathcal{B}[X] = X.  
  ]

Thus, (\mathcal{B}) is a genuine **projector** onto the hinge layer.

**5.3.3 Boundary projector on carriers**

We can specialize (\mathcal{B}) to act on radially extended carriers. Let  
[  
X(r) = \mathcal{C}^{(r)} = (k^{(r)}, h^{(r)}, \mathrm{IN}^{(r)}, \mathrm{ON}^{(r)}),  
]  
be a radially indexed family of carriers related by appropriate coarse-graining/refinement maps.

**Definition 5.3.2 (Carrier-level boundary projection).**  
The boundary projector (\mathcal{B}) applied to a radial carrier family ({\mathcal{C}^{(r)}}\_r) yields the hinge carrier  
[  
\mathcal{B}\left[{\mathcal{C}^{(r)}}\_r\right] := \mathcal{C}^{(0)}.  
]

Additionally, we can think of (\mathcal{B}) as acting on a single carrier (\mathcal{C}) that implicitly carries radial structure:

* If (\mathcal{C}) is already a hinge-level carrier (i.e., its IN/ON configuration is “reduced” to (r=0)), then  
  [  
  \mathcal{B}(\mathcal{C}) = \mathcal{C}.  
  ]
* If (\mathcal{C}) includes explicit radial substructure (e.g. nested IN/ON layers across (r)), then (\mathcal{B}(\mathcal{C})) collapses that substructure to a hinge-level effective configuration.

We do not formalize the internal coarse-graining procedure here; it is enough that such a procedure is assumed to exist and that (\mathcal{B}) is idempotent and well-defined.

**5.3.4 Boundary projector on radial functions and observables**

For scalar or vector-valued functions of (r) (e.g. dimension profiles, dual exponents, radial observables), (\mathcal{B}) acts simply by evaluation at the hinge:

* For (f(r)), we set  
  [  
  \mathcal{B}[f] := f(0).  
  ]

This is consistent with the general definition: the hinge layer is characterized by the **value** of radial functions at (r=0), and (\mathcal{B}) extracts that value.

In particular:

* For the dimension profile,  
  [  
  \mathcal{B}[D] = D(0) = 2.  
  ]
* For the present exponent,  
  [  
  \mathcal{B}[d\_{\text{PMS}}] = d\_{\text{PMS}}(0) = 2.  
  ]
* For any hinge-based observable derived from these profiles, (\mathcal{B}) simply picks out its **pivot value**.

**5.3.5 Pivot meaning: why the hinge is special**

With (\mathcal{B}) in hand, we can sharpen the idea that the hinge (r=0) is not just “one among many” radial positions, but a **pivot**:

1. **Radial collapse point**  
   (\mathcal{B}) is the operator that collapses all radial structure onto a single layer. The fact that it projects onto (r=0) and not some other value is a **structural choice** of the theory:
   * The hinge is where the present exponent (d\_{\text{PMS}}(r)) and dimension profile (D(r)) coincide at 2.
   * It is also where the pivot function (g(D)) is normalized, (g(D(0)) = g(2) = 1).
2. **Neutral band for gates and amplitudes**  
   At the pivot, the gate function is neutral:  
   [  
   g(D(0)) = 1,  
   ]  
   and the present plane structure (Section 4.4) is anchored. Projecting radially via (\mathcal{B}) brings configurations into the band where:
   * gate weights are unscaled,
   * amplitudes live in the present plane with no extra radial weights, and
   * the Born-style relation between amplitudes and IN basins is enforced in its most direct form.
3. **Double-flip + projection symmetry (preview)**  
   In the next subsection (5.4), we will state a theorem about **double-flip** sequences (e.g. two renewals or two sinks) followed by (\mathcal{B}):
   * At the pivot, the distributions resulting from different double-flip sequences become equivalent after projection by (\mathcal{B}).
   * This is one way in which the hinge acts as a fixed point of certain combinations of the operator algebra once radial structure is collapsed.

In short, the boundary projector (\mathcal{B}) is the mathematical device that **focuses** the theory onto the hinge band. The special status of (r=0) as a pivot is reflected in:

* the normalization (D(0)=2),
* the gate normalization (g(2)=1), and
* the collapse property that certain flip patterns become indistinguishable after projecting to the hinge.

**5.3.6 Summary**

Section 5.3 has:

* Introduced the **boundary projector** (\mathcal{B}) as an idempotent operator that collapses radial structure to the hinge layer (r=0).
* Defined (\mathcal{B}) on:
  + radially extended objects (families (X(r))),
  + carriers with implicit radial structure, and
  + scalar functions of (r) (via evaluation at 0).
* Clarified the **pivot meaning** of the hinge: it is the unique layer onto which radial structure is projected and at which:
  + the inner dimension (D) is 2,
  + the present exponent is 2,
  + and the pivot function (g) is normalized.
* Prepared the ground for the **double-flip + projection** symmetry, in which applying (\mathcal{B}) after certain combinations of flips yields matched distributions at the pivot.

In the next subsection (5.4), we will formulate that double-flip + projection result as a structural theorem, showing how the hinge acts as a fixed point of certain two-step operations once radial detail is collapsed by (\mathcal{B}).

**5.4 Double-Flip + Projection Theorem**

We now state a structural result that combines:

* the **radial profile** (D(r)),
* the **boundary projector** (\mathcal{B}),
* and the **flip algebra** (in particular, two-step uses of (F) and (S)),

to show that the hinge layer behaves as a **fixed point** for certain double-flip patterns, once radial detail is collapsed.

Informally, the theorem says:

At the hinge, after collapsing radial structure with (\mathcal{B}), two-step “push in” and two-step “pull out” operations yield the same effective hinge-level distribution. The hinge behaves like a balanced “sink/renew pivot” when viewed only at the present layer.

**5.4.1 Setup: radial families and flip-induced radial shifts**

Consider a radially extended description of carriers around the hinge. For concreteness, imagine:

* an initial hinge-level carrier (\mathcal{C}^{(0)}) at (r=0), and
* an extended family ({\mathcal{C}^{(r)}}\_r) such that:
  + inward refinements correspond to (\mathcal{C}^{(r)}) with (r<0),
  + outward aggregates correspond to (\mathcal{C}^{(r)}) with (r>0).

We do **not** need an explicit construction of this family; we only assume that:

* certain operator sequences tend to “push” configurations inward (toward more volume-like regimes, (r<0)),
* others tend to “pull” them outward (toward more filamentary regimes, (r>0)).

We encode this at an effective level by assigning to each primitive operator an **average radial shift**:

* (F) has a typical outward radial shift,
* (S) has a typical inward radial shift,

so that, schematically,  
[  
F : r \mapsto r + \Delta r\_F,\quad  
S : r \mapsto r + \Delta r\_S,  
]  
with (\Delta r\_F > 0), (\Delta r\_S < 0).

These shifts are understood as **effective tendencies**, not exact deterministic rules: actual radial changes may fluctuate around these averages, but the sign bias captures the net inward/outward character.

**5.4.2 Double-flip operations around the hinge**

We focus on **two-step** words built from the same primitive:

* (FF) (two renew steps), and
* (SS) (two sink steps).

From the radial-shift viewpoint, these behave as:

* (FF): net outward shift (\approx 2\Delta r\_F > 0),
* (SS): net inward shift (\approx 2\Delta r\_S < 0).

Starting from a hinge-level carrier (\mathcal{C}^{(0)}), the effective radial positions reached after these double flips are:

* (FF)-path: some distribution of carriers (X\_F) concentrated around a radial level (r\_F > 0),
* (SS)-path: some distribution of carriers (X\_S) concentrated around a radial level (r\_S < 0).

We will not try to track the full shape of these radial distributions; the key step is what happens when we **project them back** to the hinge via (\mathcal{B}).

**5.4.3 Projection to the hinge and dimension symmetry**

Recall that the boundary projector (\mathcal{B}) collapses radial structure to the hinge layer:

* (\mathcal{B}) acting on a radially extended object returns its effective hinge-level representative,
* As a result, any information that depends only on the hinge dimension and hinge-level IN geometry (e.g. the present plane, gate, and Born-style weights) is read from (\mathcal{B}[X]).

The dimension profile (D(r)) has hinge value (D(0)=2), and, in many constructions, satisfies an approximate mirror behavior near the hinge (for small (|r|)):

* inward shifts and outward shifts of equal magnitude move (D(r)) away from 2 by similar amounts in opposite directions.

When we apply (\mathcal{B}), we are effectively:

* “forgetting” whether the configuration came from inward or outward radial displacement, and
* retaining only the **hinge-level IN structure** and associated hinge observables.

The double-flip + projection theorem formalizes the idea that, under mild symmetry assumptions near the hinge, **two-step inward** and **two-step outward** flips produce hinge-level distributions that are indistinguishable once (\mathcal{B}) has been applied.

**5.4.4 Double-flip + projection theorem**

We now state the result in a generic form.

**Theorem 5.4.1 (Double-flip + projection equivalence at the hinge).**

Let (\mathcal{C}^{(0)}) be an admissible hinge-level carrier at (r=0). Consider:

* The family of carriers (X\_F) reachable from (\mathcal{C}^{(0)}) by all admissible two-step words equal to (FF) up to neutral moves:  
  [  
  X\_F := { \Pi(\mathcal{C}^{(0)}) \mid \Pi \approx FF,; \Pi \text{ admissible} }.  
  ]
* The family of carriers (X\_S) reachable from (\mathcal{C}^{(0)}) by all admissible two-step words equal to (SS) up to neutral moves:  
  [  
  X\_S := { \Pi(\mathcal{C}^{(0)}) \mid \Pi \approx SS,; \Pi \text{ admissible} }.  
  ]

Assume:

1. **Radial symmetry near the hinge:**  
   For small radial shifts (\pm \Delta r), the IN dimension and related hinge observables satisfy  
   [  
   D(+\Delta r) \approx D(-\Delta r),  
   ]  
   and other hinge-level structural functions (e.g. those used in gate and Born-style rules) are likewise approximately symmetric under (r \to -r) near (0).
2. **Balanced double shifts:**  
   Typical radial shifts induced by (FF) and (SS) starting from (r=0) are symmetric in magnitude:  
   [  
   r\_F \approx +\Delta r,\quad r\_S \approx -\Delta r,  
   ]  
   for some small (\Delta r>0), modulo fluctuations that average out over the ensemble of admissible paths.

Then, after applying the boundary projector (\mathcal{B}), the resulting **hinge-level distributions** are equivalent in the following sense:

* The distributions of hinge-level carriers (\mathcal{B}[X\_F]) and (\mathcal{B}[X\_S]) match for all observables that depend only on hinge-level IN structure and present-plane amplitudes. In particular, for any hinge-level observable (O\_{\text{hinge}}) built from:
  + the hinge IN attractor (dimension (D(0)=2)),
  + the present-plane amplitudes and Born-style weights, and
  + **the pivot function (g(D)) evaluated at (D=2),  
    we have  
    [  
    \mathbb{E}*{\mathcal{B}[X\_F]}[O*{\text{hinge}}]**

\mathbb{E}*{\mathcal{B}[X\_S]}[O*{\text{hinge}}],  
]  
where (\mathbb{E}) denotes expectation over the respective ensembles.

In words: **two-step renewal and two-step sinking look the same at the hinge once radial detail is projected away.**

*Sketch of reasoning.*

* The families (X\_F) and (X\_S) differ primarily in **radial displacement direction**: outward vs inward.
* Near the hinge, the dimension profile and other structural functions are assumed radially symmetric (or nearly so), so that:  
  [  
  D(r\_F) \approx D(r\_S),\quad g(D(r\_F)) \approx g(D(r\_S)),\quad \text{etc.}  
  ]
* Applying (\mathcal{B}) collapses these configurations back to the hinge, where they share:
  + the same IN dimension (D(0)=2),
  + the same gate normalization (g(2)=1),
  + and the same present-plane structure.
* Any observable that depends only on the resulting hinge-level configuration (not on the path taken in radial space) therefore cannot distinguish whether the underlying configuration came from an (FF)-type or (SS)-type displacement.

This is an **approximate** equality that becomes exact in the idealized limit of perfect radial symmetry near the hinge and negligible higher-order corrections.

(\square)

**5.4.5 Interpretation and role in the theory**

The double-flip + projection theorem highlights the special status of the hinge:

1. **Sink–renew balance at the pivot.**  
   Although (F) and (S) are not inverse operations and drive the ledger in different directions, when we look only at two-step operations and then collapse radial structure, certain combinations become **indistinguishable** at the hinge. The hinge thus behaves as a **balanced point** where:
   * inward and outward excursions of equal magnitude are symmetrically encoded, and
   * the present layer “forgets” which direction the radial excursion took, once we project back with (\mathcal{B}).
2. **Robustness of hinge-level observables.**  
   Many core observables of the theory (present-plane amplitudes, Born-style weights, gate-neutral couplings) live at the hinge layer. The theorem indicates that these observables are robust against certain symmetric patterns of sink and renew: their statistics do not depend on whether the underlying configuration briefly moved inward or outward in a balanced way.
3. **Preparation for pivot identities.**  
   In later parts of the theory (especially when we introduce explicit S² collapse kernels), this kind of **hinge invariance under double operations** will reappear in more concrete forms, such as:
   * collapse kernels becoming identity at (D=2),
   * mirror-symmetry functionals singling out the hinge as a pivot, and
   * unified inverse-square laws emerging from sphere-like behavior at the hinge.

The double-flip + projection theorem is thus the **radial counterpart** of those later pivot identities: it shows, in the simplest operator-algebraic setting, how the hinge washes out the memory of symmetric inward/outward deviations when viewed solely from the present layer.

**5.4.6 Summary**

Section 5.4 has:

* Defined the setting for **two-step** flip sequences (FF, SS) around the hinge and their radial displacements.
* Introduced symmetry assumptions on the radial profile and radial shifts near the hinge.
* Stated the **Double-flip + Projection Theorem**, which asserts that, after applying the boundary projector (\mathcal{B}), the hinge-level distributions resulting from FF and SS paths are equivalent for all observables that depend only on hinge-level structure.
* Interpreted this equivalence as a sign that the hinge is a **sink–renew pivot**: a point where symmetric inward/outward excursions look the same once radial detail is collapsed.

In the next subsection (5.5), we will use this pivot behavior to place **cross-scale constraints on the pivot function (g(D))**, ensuring that its dependence on (D) is compatible with the hinge’s role as a symmetric interface between inner (volume-like) and outer (filamentary) regimes.

**5.5 Cross-Scale Constraints on (g(D))**

The radial profile (D(r)) describes how the effective IN dimension changes across contexts; the pivot function (g(D)) provides a scalar weight that modulates couplings and action contributions at each dimension. In this section we state the **cross-scale constraints** on (g(D)) required for:

* compatibility with the hinge behavior of (D(r)),
* well-behaved sums and integrals over context bands, and
* symmetric treatment of inner/outer deviations around the pivot.

We remain deliberately non-committal about the exact analytic form of (g(D)); only its **structure** is fixed.

**5.5.1 Local hinge constraints**

Recall:

* The dimension profile is (D(r) = 2 + \delta(r)), with (D(0)=2).
* The pivot function satisfies (g(2)=1) and is continuous on ([D\_{\min},D\_{\max}]).

Near the hinge, we require that (g(D)) behave smoothly and that its departures from 1 be controlled.

**Axiom 5.5.1 (Local hinge regularity).**

There exists an interval ([2-\varepsilon,2+\varepsilon]) such that:

1. (g) is (C^1) (continuously differentiable) on this interval.
2. The derivative at the hinge is finite:  
   [  
   |g'(2)| < \infty.  
   ]
3. For all (D) in this interval,  
   [  
   |g(D) - 1| \le L, |D-2|  
   ]  
   for some Lipschitz constant (L > 0).

This guarantees that:

* small shifts in dimension near the hinge produce only small, controlled changes in the gate weight, and
* the hinge (D=2) is a **stable reference point** for perturbative constructions (e.g., expansions of the master action around the pivot).

In many symmetric realizations, one further imposes (g'(2)=0), making (D=2) a local extremum of (g); this is optional for the core formalism but often convenient.

**5.5.2 Mirror behavior under inward/outward deviations**

The dimension profile (D(r)) distinguishes inward ((r<0)) and outward ((r>0)) contexts, but the hinge is treated as a **balance point**. For small (|r|), we want the gate weight to respond symmetrically (or nearly so) to equal-magnitude inward and outward deviations.

Let (\delta(r) = D(r) - 2). For (|r|) sufficiently small, we have (D(r) \in (2-\varepsilon,2+\varepsilon)) and can compose (g) with (D(r)).

**Axiom 5.5.2 (Approximate mirror symmetry near the hinge).**

There exists (\varepsilon>0) and a small error function (\eta(r)) (with (\eta(0)=0)) such that, for all (|r|\le\varepsilon),  
[  
g(D(r)) = g(D(-r)) + \eta(r),  
]  
with (|\eta(r)|) bounded and (|\eta(r)| \ll |g(D(r)) - 1|) in that range.

In the idealized mirror-symmetric case, (\eta(r) \equiv 0) and we have  
[  
g(D(r)) = g(D(-r)).  
]

This is the **gate analogue** of the radial symmetry assumptions used in the double-flip + projection theorem: equal-magnitude inward and outward shifts from the hinge carry the same gate weight.

**5.5.3 Asymptotic behavior and boundedness across bands**

As we move far inward or outward, the dimension (D(r)) approaches limiting values:

* inward: (D(r)\to D\_{\text{in}} > 2),
* outward: (D(r)\to D\_{\text{out}} < 2).

We want (g(D)) to remain **bounded** and **non-singular** across the entire allowed dimension interval.

**Axiom 5.5.3 (Global boundedness).**

There exist constants (0 < g\_{\min} \le g\_{\max} < \infty) such that  
[  
g\_{\min} \le g(D) \le g\_{\max}  
]  
for all (D \in [D\_{\min},D\_{\max}]).

This ensures that:

* no context band has **infinite** or **zero** coupling strength in the pure theory, and
* sums or integrals over bands weighted by (g(D)) (e.g., in the master action) are well-defined and do not diverge solely due to gate weights.

We may also require:

* **tame asymptotic behavior**:  
  [  
  \lim\_{D\to D\_{\text{in}}} g(D) = g\_{\text{in}},\quad  
  \lim\_{D\to D\_{\text{out}}} g(D) = g\_{\text{out}},  
  ]  
  with (g\_{\text{in}},g\_{\text{out}}) finite and within ([g\_{\min},g\_{\max}]).

**5.5.4 Compatibility with the master action**

In the discrete ladder (Section 7), each band (n) has dimension (D(n)) and contributes to the **master action** with a weight (g(D(n))). Let:

* (n\in\mathbb{Z}) index context bands,
* (D(n) = D(r(n))) be the bandwise dimension profile,
* (g\_n := g(D(n))) be the bandwise gate weight.

The master action typically contains sums or integrals of the form  
[  
S\_{\text{disc}} \sim \sum\_{n} g\_n, \Delta S\_n,  
]  
or, in a continuum limit,  
[  
S\_{\text{cont}} \sim \int g(D(r)), \mathcal{L}(r), \mathrm{d}r,  
]  
where (\Delta S\_n) and (\mathcal{L}(r)) are bandwise or radial contributions.

To avoid pathological behavior, we require:

**Axiom 5.5.4 (Action-compatibility).**

1. (g(D)) is measurable and bounded on ([D\_{\min},D\_{\max}]), so that band-sums and radial integrals weighted by (g(D)) are finite whenever the underlying (\Delta S\_n) or (\mathcal{L}(r)) are.
2. There are no **essential singularities** of (g) within ([D\_{\min},D\_{\max}]); any sharp changes must occur only at the endpoints (and even there they are bounded).
3. For any finite radial or band interval, the variation of (g(D)) is bounded:  
   [  
   |g(D\_1) - g(D\_2)| \le L\_{I}, |D\_1 - D\_2|  
   ]  
   for some interval-dependent Lipschitz constant (L\_I). This ensures stability of the action under small changes in (D).

These conditions guarantee that (g(D)) can be safely inserted into the dynamical layer without introducing inconsistencies or divergences.

**5.5.5 Cross-scale reciprocity**

Finally, we encode a **reciprocity condition** that reflects the idea that inner and outer contexts are two sides of a single radial structure, and their coupling weights should be related in a simple way.

One natural structural choice is that inner and outer limiting values of (g(D)) satisfy a reciprocal relation, for example:  
[  
g\_{\text{in}} , g\_{\text{out}} \approx 1,  
]  
or, more generally,  
[  
g(D\_{\text{in}}), g(D\_{\text{out}}) \approx g(2)^2 = 1.  
]

We do not enforce a strict equality in the core formalism, but we **allow** such reciprocal forms when we want a more constrained model. The general statement is:

**Axiom 5.5.5 (Cross-scale reciprocity, optional strengthening).**

In models that implement explicit reciprocity between inner and outer regimes, (g(D)) is chosen such that for appropriately defined inner and outer limiting dimensions, a simple algebraic relation (such as reciprocal or symmetric behavior) holds:  
[  
\Phi\bigl(g(D\_{\text{in}}), g(D\_{\text{out}})\bigr) = \Phi(1,1),  
]  
for some symmetric function (\Phi). The hinge value (g(2)=1) is then the mediating point of this reciprocity.

This captures the conceptual idea that the hinge connects an “inside” and an “outside” whose coupling strengths are balanced around the pivot.

**5.5.6 Summary**

Section 5.5 has:

* Imposed **local hinge constraints** on (g(D)): continuity, differentiability, Lipschitz behavior, and controlled deviations from the normalized value (g(2)=1).
* Required **approximate mirror symmetry** of (g(D(r))) under (r \to -r) near the hinge, aligning the gate behavior with inward/outward radial symmetries.
* Enforced **global boundedness** and tameness of (g(D)) across the full dimension interval ([D\_{\min},D\_{\max}]), preventing singularities or diverging weights.
* Specified **action-compatibility** conditions so that (g(D))-weighted band contributions yield well-behaved master actions.
* Allowed for **cross-scale reciprocity** between inner and outer regimes, with the hinge normalization (g(2)=1) as a central reference.

These cross-scale constraints ensure that the pivot function (g(D)) is structurally consistent with the radial dimension profile, the hinge properties, and the dynamical layer of the theory. In the next subsection (5.6), we will move from this continuous radial picture (D(r)) to the **discrete ladder** (D(n)), introducing the nested (\mathrm{CS}\_n) hierarchy and the notion of fractal reciprocity across context bands.

**5.6 Nested (\mathrm{CS}\_n) Ladder & Fractal Reciprocity**

Sections 5.1–5.5 described the **radial** picture: a continuous parameter (r) labelling inner and outer contexts, with a dimension profile (D(r)), hinge projector (\mathcal{B}), and cross-scale constraints on (g(D)). We now translate this into a **banded** picture built from **nested Collective Spheres** (\mathrm{CS}\_n), and state the notion of **fractal reciprocity** between inner ((D>2)) and outer ((D<2)) regimes.

This prepares the ground for the fully discrete context ladder ((D(n), g(D(n)))) and the reproduction kernel in Part VI.

**5.6.1 From radial parameter (r) to band index (n)**

We introduce an integer index (n \in \mathbb{Z}) to label discrete **context bands** that approximate radial levels:

* (n=0): hinge band, corresponding to (r=0).
* (n<0): inner bands (finer contexts), corresponding to (r<0).
* (n>0): outer bands (coarser contexts), corresponding to (r>0).

We assume:

* There exists a **monotone mapping** (n \mapsto r(n)) such that:
  + (r(0) = 0),
  + (n\_1 < n\_2 \Rightarrow r(n\_1) < r(n\_2)).

We then define the **bandwise dimension profile** by sampling the radial profile:  
[  
D(n) := D\bigl(r(n)\bigr),  
]  
with  
[  
D(0) = D(r(0)) = 2.  
]

Likewise, we set the **bandwise pivot weights**:  
[  
g\_n := g\bigl(D(n)\bigr).  
]

This gives a discrete ladder ({D(n),g\_n}\_{n\in\mathbb{Z}}) that mirrors the continuous profile (D(r), g(D(r))).

**5.6.2 Nested (\mathrm{CS}\_n) hierarchy**

At each band (n), we can consider Collective Spheres that are **native** to that band.

**Definition 5.6.1 (Bandwise Collective Spheres).**  
For each (n \in \mathbb{Z}), a **band-(n) Collective Sphere** (\mathrm{CS}\_n) is a synchronized collection of PMSs whose IN/ON structure is described at band (n) and whose effective IN dimension is (D(n)).

We assume:

* For each band (n), there is at least one admissible (\mathrm{CS}\_n).
* Bandwise CSs form a **nested hierarchy** in the sense that:
  + an (\mathrm{CS}*n) is “contained within” an (\mathrm{CS}*{n+1}) as an inner subsystem,
  + and “contains” (\mathrm{CS}\_{n-1}) as an inner refinement.

This nesting is not necessarily literal set inclusion; it is a **structural inclusion**:

* The boundary configuration of (\mathrm{CS}*n) is compatible with a coarse-grained view of the boundary of (\mathrm{CS}*{n-1}) and a refined view of that of (\mathrm{CS}\_{n+1}).
* IN/ON relationships at neighboring bands are linked by coarse-graining and refinement operators (formalized later with the reproduction kernel).

At the hinge, (\mathrm{CS}\_0) is the band whose IN dimension is (D(0)=2) and whose gate weight is normalized, (g\_0 = g(2) = 1).

**5.6.3 Bandwise pivot weights and local couplings**

Each (\mathrm{CS}\_n) carries its own pivot weight (g\_n) and thus its own **effective coupling strength**:

* Within (\mathrm{CS}\_n): coupling and interaction strengths are scaled by (g\_n).
* Between (\mathrm{CS}\_n) and (\mathrm{CS}\_m): cross-context weights are given by a symmetric combination of (g\_n) and (g\_m) (e.g. (\min(g\_n,g\_m))).

This yields a **ladder of frames**:

* Each (\mathrm{CS}\_n) acts as a frame with its own internal interval structure (inherited from the flip algebra) and its own gate scaling via (g\_n).
* The hinge frame (\mathrm{CS}\_0) is the reference; other frames are compared to it via the bandwise weights.

**5.6.4 Fractal reciprocity between inner and outer bands**

The dimension profile satisfies (D(n) > 2) for sufficiently negative (n) (inner bands) and (D(n) < 2) for sufficiently positive (n) (outer bands). We call this **fractal reciprocity**:

Inner bands are “thicker” (more volume-like) and outer bands are “thinner” (more filamentary), but they are related through a hinge band at (D=2) that mediates their mutual influence.

Formally, we encode reciprocity via:

**Definition 5.6.2 (Fractal reciprocity across bands).**

The ladder ({D(n)}) is said to satisfy **fractal reciprocity** if there exists a mapping (n \mapsto \bar{n}) (an “outer partner” index) such that:

1. (\bar{\bar{n}} = n) (involution),
2. (D(n) > 2 \iff D(\bar{n}) < 2),
3. For bands paired under this map, deviations from the hinge are related in a simple way, e.g.  
   [  
   D(n) - 2 \approx -\bigl(D(\bar{n}) - 2\bigr),  
   ]  
   or more generally a symmetric relation compatible with the radial mirror behavior.

In a perfectly mirror-symmetric ladder, one could choose (\bar{n} = -n) and obtain:  
[  
D(-n) + D(n) = 4 \quad \text{for all } n,  
]  
though the formalism only requires approximate symmetry near the hinge and well-behaved asymptotics.

This reciprocity expresses:

* inward bands ((n<0)) as **reciprocal partners** of outward bands ((n>0)),
* with the hinge band (n=0) acting as the **central pivot** that connects the two regimes.

**5.6.5 Nested ladder and present-moment structure**

The nested (\mathrm{CS}\_n) ladder provides a multi-scale view of the present-moment structure:

* At band (n=0), (\mathrm{CS}\_0) captures the present environment itself (hinge band).
* Bands (n=-1,-2,\dots) describe inward nested structures (subsystems, finer internal organization) that are **experienced as “inside”** the present environment.
* Bands (n=+1,+2,\dots) describe outward nested structures (larger-scale environments, ambient fields) that are **experienced as “around” or “containing”** the present.

Each band has:

* its own IN dimension (D(n)) and gate weight (g\_n),
* its own CSs as frames,
* and connections to adjacent bands via coarse-graining/refinement, respecting the fractal reciprocity conditions.

Thus, a single “present moment” can be seen as a **stack of nested contexts** ({\mathrm{CS}\_n}), all anchored on the hinge band.

**5.6.6 Summary**

Section 5.6 has:

* Introduced the discrete **band index** (n) as a sampled version of the radial parameter (r), with (D(n) = D(r(n))) and (D(0)=2).
* Defined bandwise Collective Spheres (\mathrm{CS}\_n) forming a **nested hierarchy** across inner and outer contexts.
* Attached bandwise pivot weights (g\_n = g(D(n))) that scale couplings and action contributions at each band.
* Stated **fractal reciprocity**: inner bands (with (D>2)) and outer bands (with (D<2)) are related through a hinge band at (D=2), often approximately mirror-symmetric under (n \leftrightarrow \bar{n}).
* Interpreted the nested ({\mathrm{CS}\_n}) ladder as a multi-scale present-moment structure, with hinge (\mathrm{CS}\_0) at its center.

With this, the continuous radial picture (Sections 5.1–5.5) has been translated into a discrete, banded ladder with nested CSs and fractal reciprocity. In Part VI (Section 6), we will build on this ladder to introduce the **multi-context graph**, the **reproduction kernel**, and the explicit discrete dimension curve (D(n)) used in the master action and quantization.

**6. Discrete Context Ladder, Reproduction Kernel & (D(n))**

**6.1 Multi-Context Graph & Boundary Graphs**

In Part V we described a **continuous radial picture** of contexts using a parameter (r) and a dimension profile (D(r)). We now move to a **discrete ladder** description indexed by integers (n \in \mathbb{Z}). This section introduces:

* the **multi-context graph** that organizes context bands and their relations, and
* the **boundary graphs** associated with each band, which will be the natural home for gauge and collapse structures later on.

**6.1.1 Context bands and ladder index (n)**

We fix a discrete context index set  
[  
N := { n \mid n \in \mathbb{Z} },  
]  
with elements called **bands** or **context levels**. As in Section 5.6:

* (n = 0) is the **hinge band**,
* (n < 0) are **inner bands**,
* (n > 0) are **outer bands**.

We assume there is a monotone map (n \mapsto r(n)) from band indices to radial coordinates such that:

* (r(0) = 0),
* (n\_1 < n\_2 \Rightarrow r(n\_1) < r(n\_2)),

and define the bandwise dimension profile by  
[  
D(n) := D\bigl(r(n)\bigr), \quad D(0) = 2.  
]

The goal of this section is to attach a **graph structure** to these bands that encodes:

* how bands are connected to each other (up/down in (n)), and
* how the **boundary** of each band is organized internally.

**6.1.2 Multi-context graph (\mathcal{G})**

We first introduce an abstract graph that organizes all bands and their boundary structures.

**Definition 6.1.1 (Multi-context graph).**  
The **multi-context graph** (\mathcal{G}) is a layered graph built from:

1. A set of **band nodes**  
   [  
   { b\_n \mid n \in N },  
   ]  
   one for each context level (n).
2. For each band (n), a **boundary graph** (\mathcal{G}\_n) (defined in 6.1.3), whose nodes and edges represent the internal boundary structure of that band.
3. **Inter-band edges** connecting neighboring bands, which encode coarse-graining/refinement relationships between (\mathcal{G}*n) and (\mathcal{G}*{n\pm 1}).

Formally,  
[  
\mathcal{G} = \bigl( V, E \bigr),  
]  
where:

* (V) is the disjoint union of band nodes and boundary graph nodes:  
  [  
  V = {b\_n : n\in N} ;\sqcup; \bigsqcup\_{n\in N} V\_n,  
  ]  
  with (V\_n) the vertex set of (\mathcal{G}\_n).
* (E) consists of:
  + **intra-band edges** within each boundary graph (\mathcal{G}\_n), and
  + **inter-band edges** connecting nodes in (V\_n) to nodes in (V\_{n\pm 1}), representing coarse-grain/refinement adjacency.

We do not fix a particular combinatorial structure for (\mathcal{G}); rather, we require only:

* each (\mathcal{G}\_n) is a connected graph (boundary is “all of one piece” at each band),
* each node in (V\_n) connects to at least one node in (V\_{n-1}) and one in (V\_{n+1}), so that inner and outer adjacency is represented.

**6.1.3 Boundary graphs (\mathcal{G}\_n)**

The boundary graph at band (n) captures the connectivity of the band’s **present boundary**—the abstract analogue of a surface at that context level.

**Definition 6.1.2 (Boundary graph at band (n)).**  
For each band (n), the **boundary graph** (\mathcal{G}\_n = (V\_n, E\_n)) is a finite or countable graph whose:

* vertices (V\_n) represent **boundary patches** or **boundary sites** at context level (n),
* edges (E\_n) represent adjacency relations between boundary patches (e.g. which patches share a common interface or can exchange boundary data via operator action).

We assume:

1. (\mathcal{G}\_n) is **connected**.
2. The combinatorial structure of (\mathcal{G}\_n) reflects the bandwise IN dimension (D(n)) in the sense that:
   * for large subgraphs, the growth of reachable nodes as a function of graph radius is consistent with an effective dimension (D(n)) (e.g. doubling radius increases node count approximately by a factor (2^{D(n)}) in appropriate regimes).

This is an abstract way of saying: the boundary at band (n) is “surface-like” with effective dimension (D(n)), without committing to a particular geometry.

**6.1.4 Band-to-band adjacency**

We now specify how boundary graphs at adjacent bands are linked.

**Definition 6.1.3 (Inter-band adjacency).**  
For each pair of neighboring bands (n, n+1), we introduce a set of **inter-band edges**  
[  
E\_{n,n+1} \subseteq V\_n \times V\_{n+1},  
]  
such that:

1. Every vertex (v \in V\_n) has at least one neighbor in (V\_{n+1}) and at least one neighbor in (V\_{n-1}) (except at any imposed outer limits, if they exist).
2. The inter-band edges represent **coarse-graining/refinement relations**:
   * an edge ((v\_n, v\_{n+1}) \in E\_{n,n+1}) indicates that the boundary patch (v\_n) at band (n) is refined into, or arises as a coarse-graining of, the patch (v\_{n+1}) at band (n+1).

These inter-band edges are the discrete analogue of the radial coupling between levels (r) and (r\pm \Delta r). They will be used to define:

* **collapse operators** that map from fine to coarse bands (and vice versa), and
* the **reproduction kernel** that describes how IN structure at one band generates or is generated by IN structure at neighboring bands.

**6.1.5 Hinge boundary graph (\mathcal{G}\_0) and pivot role**

The band (n=0) is the hinge band where (D(0)=2), and its boundary graph (\mathcal{G}\_0) plays a special role.

We treat (\mathcal{G}\_0) as the **pivot boundary graph**:

* it is the graph on which:
  + the hinge-level collapse kernel (later in Section 11) acts as a constant projector,
  + the pivot function (g(D)) is normalized (so (g\_0 = g(D(0)) = 1)),
  + and the present-plane amplitudes and Born-style weights are naturally attached to nodes or regions.

Formally:

* (\mathcal{G}\_0) is a distinguished boundary graph with effective dimension (D(0)=2), and
* all other (\mathcal{G}\_n) are connected to (\mathcal{G}\_0) through inter-band edges such that the hinge band sits between “inner” (D>2) and “outer” (D<2) boundary structures.

Geometrically, one can think of (\mathcal{G}\_0) as playing a role similar to a **2-sphere boundary** in a continuous setting, without forcing an exact topological S² structure at this discrete, abstract level.

**6.1.6 Context graph for carriers**

Finally, we relate the multi-context graph (\mathcal{G}) to **carriers** and **Collective Spheres**.

* Each vertex (v\_n \in V\_n) can be associated with a **boundary patch** within a band-(n) CS, and thus with a family of carriers whose IN/ON structure is localized on that patch at band (n).
* A **band-(n) CS** can then be viewed as:
  + a meta-node associated with the entire boundary graph (\mathcal{G}\_n), and
  + a collection of carriers whose boundary states populate (V\_n) in a synchronized way.

The multi-context graph (\mathcal{G}) thus serves as:

* a **scaffolding** on which we can track how boundary configurations at different bands are related, and
* a **domain** for operators that move structure up and down the ladder (collapse/expansion) and across boundary adjacency within each band.

In subsequent sections:

* Section 6.2 will define a **collapse & expansion algebra** on this ladder (operators mapping between (\mathcal{G}*n) and (\mathcal{G}*{n\pm1})),
* Section 6.3 will introduce the **reproduction kernel** (M) and the associated **memory dimension** (D\_{\mathrm{mem}}(n)), and
* Section 6.5 will specify a concrete form for the **discrete dimension curve** (D(n)) consistent with the radial profile and hinge behavior.

**6.1.7 Summary**

Section 6.1 has:

* Introduced the **multi-context graph** (\mathcal{G}) as a layered graph over bands (n\in\mathbb{Z}), with band nodes and boundary graphs (\mathcal{G}\_n).
* Defined **boundary graphs** (\mathcal{G}\_n) as connected graphs whose large-scale combinatorics reflect the bandwise IN dimension (D(n)).
* Specified **inter-band edges** that encode coarse-graining/refinement relations between neighboring bands.
* Identified the hinge boundary graph (\mathcal{G}\_0) as a pivot layer with dimension (D(0)=2) and normalized gate weight (g\_0=1).
* Explained how carriers and CSs can be organized over (\mathcal{G}), setting the stage for ladder-level collapse and reproduction dynamics.

In the next subsection (6.2), we will define the **collapse and expansion operators** that act on this multi-context graph, providing the algebra that moves structures up and down the discrete ladder.

**6.2 Collapse & Expansion Algebra on the Ladder**

In Section 6.1 we introduced the multi-context graph (\mathcal{G}) with boundary graphs (\mathcal{G}\_n) at each band (n). We now define an algebra of **collapse** (downward) and **expansion** (upward) maps that move boundary states between neighboring bands:

* Collapse maps coarse-grain from band (n+1) down to (n),
* Expansion maps refine from band (n) up to (n+1).

These maps will be the backbone for the **reproduction kernel** and the **memory dimension** (D\_{\mathrm{mem}}(n)) in the next subsection.

**6.2.1 Bandwise boundary state spaces**

For each band (n), we associate a state space for boundary configurations on (\mathcal{G}\_n).

**Definition 6.2.1 (Boundary state space at band (n)).**  
Let (\mathcal{G}\_n = (V\_n,E\_n)) be the boundary graph at band (n). The **boundary state space** (\mathcal{H}\_n^\partial) is an abstract vector space of functions or configurations “living on” (V\_n). For example, one can think of:

* states (\psi\_n : V\_n \to \mathbb{C}), or
* more general structured states attached to nodes and edges of (\mathcal{G}\_n).

We do **not** fix the precise nature of (\mathcal{H}\_n^\partial); we only assume:

* it is a complex (or real) vector space,
* it admits linear maps induced by graph operations, and
* its dimension or effective degrees of freedom reflect the combinatorics of (\mathcal{G}\_n) and the band’s IN dimension (D(n)).

Boundary-level descriptions of carriers and CSs at band (n) can then be represented as vectors in (\mathcal{H}\_n^\partial).

**6.2.2 Collapse operators (K\_{n+1\to n})**

A **collapse operator** maps boundary states on a finer (or more outer) band down to a coarser (or more inner) band, implementing a bandwise coarse-graining.

**Definition 6.2.2 (Collapse operator).**  
For each neighboring pair of bands (n, n+1), a **collapse operator**  
[  
K\_{n+1\to n} : \mathcal{H}\_{n+1}^\partial \to \mathcal{H}\_n^\partial  
]  
is a linear map satisfying:

1. **Coarse-graining / many-to-one behavior**  
   (K\_{n+1\to n}) is typically **surjective**: every state in (\mathcal{H}*n^\partial) can be obtained as the collapse of at least one state in (\mathcal{H}*{n+1}^\partial).
2. **Graph compatibility**  
   (K\_{n+1\to n}) respects the inter-band adjacency encoded in (E\_{n,n+1}) (Definition 6.1.3). Intuitively, it combines or averages states on nodes of (V\_{n+1}) that are “grouped” together by edges into nodes of (V\_n).
3. **Band-limited behavior**  
   (K\_{n+1\to n}) does not introduce support on graph nodes that were not already connected by (E\_{n,n+1}); it only aggregates information along existing inter-band adjacency.

These operators implement a **downward** flow of boundary information along the ladder, from (n+1) to (n).

**6.2.3 Expansion operators (E\_{n\to n+1})**

Expansion operators go in the opposite direction: they lift band-(n) boundary states to band (n+1), adding degrees of freedom in a controlled way.

**Definition 6.2.3 (Expansion operator).**  
For each neighboring pair of bands (n, n+1), an **expansion operator**  
[  
E\_{n\to n+1} : \mathcal{H}*n^\partial \to \mathcal{H}*{n+1}^\partial  
]  
is a linear map satisfying:

1. **Refinement / one-to-many embedding**  
   (E\_{n\to n+1}) is typically **injective**: distinct states in (\mathcal{H}*n^\partial) remain distinct when expanded to (\mathcal{H}*{n+1}^\partial).
2. **Graph compatibility**  
   (E\_{n\to n+1}) distributes the state on band (n) across nodes in (V\_{n+1}) that are connected to the corresponding nodes in (V\_n) by edges in (E\_{n,n+1}).
3. **Band-limited behavior**  
   (E\_{n\to n+1}) does not create support on nodes in (V\_{n+1}) that are not linked (directly or indirectly) to nodes in (V\_n).

These operators implement an **upward** refinement of boundary information along the ladder.

**6.2.4 Approximate inverse and projection/embedding structure**

Collapse and expansion operators are not mutual inverses, because collapse generally **loses** information. However, we impose a standard projection/embedding relationship.

**Axiom 6.2.4 (Projection/embedding relation).**

For each neighboring pair ((n,n+1)):

1. The composite  
   [  
   K\_{n+1\to n} \circ E\_{n\to n+1} = \mathrm{id}\_{\mathcal{H}\_n^\partial}  
   ]  
   is the identity on band-(n) boundary states.
   * Interpretation: if we expand a band-(n) state to band (n+1) and then collapse it back, we recover the original band-(n) state exactly. Expansion therefore embeds (\mathcal{H}*n^\partial) into (\mathcal{H}*{n+1}^\partial) in a way that is **left-invertible** by collapse.
2. The composite  
   [  
   P\_{n+1} := E\_{n\to n+1} \circ K\_{n+1\to n}  
   ]  
   is a **projection** on (\mathcal{H}*{n+1}^\partial), i.e.  
   [  
   P*{n+1}^2 = P\_{n+1}.  
   ]
   * Interpretation: if we collapse a band-(n+1) state down to band (n) and then re-expand it, we obtain the “band-(n) component” of the original band-(n+1) state. Applying this projection twice has no further effect.

Thus, (\mathcal{H}*n^\partial) is embedded as a distinguished subspace of (\mathcal{H}*{n+1}^\partial), and collapse is the projection onto this subspace.

**6.2.5 Multi-step collapse and expansion**

We extend the definitions to non-neighboring bands via composition.

**Definition 6.2.5 (Multi-step collapse and expansion).**

For any pair of bands (m > n), define the **multi-step collapse** operator  
[  
K\_{m\to n} := K\_{n+1\to n} \circ K\_{n+2\to n+1} \circ \dots \circ K\_{m\to m-1} :  
\mathcal{H}\_m^\partial \to \mathcal{H}\_n^\partial.  
]

For any pair (m < n), define the **multi-step expansion** operator  
[  
E\_{m\to n} := E\_{n-1\to n} \circ \dots \circ E\_{m\to m+1} :  
\mathcal{H}\_m^\partial \to \mathcal{H}\_n^\partial.  
]

These maps satisfy:

* (K\_{n\to n} = \mathrm{id}\_{\mathcal{H}\_n^\partial}),
* (E\_{n\to n} = \mathrm{id}\_{\mathcal{H}\_n^\partial}), and
* for (m\le k\le n),  
  [  
  K\_{n\to m} = K\_{k\to m} \circ K\_{n\to k},\quad  
  E\_{m\to n} = E\_{k\to n} \circ E\_{m\to k}.  
  ]

Under the projection/embedding axiom, we also have:

* (K\_{n\to m} \circ E\_{m\to n} = \mathrm{id}\_{\mathcal{H}\_m^\partial}) for (n>m),
* (E\_{m\to n} \circ K\_{n\to m}) is a projection on (\mathcal{H}\_n^\partial) onto the embedded image of (\mathcal{H}\_m^\partial).

Thus the ladder supports a **hierarchy of embeddings** of inner bands into outer ones and **projections** of outer bands onto inner ones.

**6.2.6 Neutrality and ladder symmetries**

Certain composites of collapse and expansion maps can be treated as **neutral moves** in the ladder, in the sense that they do not change the effective band-level state up to the identification of embedded subspaces.

For example:

* For a state (\psi\_n \in \mathcal{H}*n^\partial),  
  [  
  E*{n\to n+1}(\psi\_n) \quad\text{and}\quad  
  P\_{n+1}(\phi\_{n+1})  
  ]  
  represent different ways of embedding or projecting states at band (n+1) that, when collapsed back to band (n), yield the same (\psi\_n).

We can therefore extend the notion of **neutral words** (Section 2.4) to include:

* ladder-level composites that leave a given band-level state invariant when viewed at that band, even if they temporarily move the state up or down the ladder.

Formally, a ladder word (W) built from (K) and (E) is **neutral at band (n)** if  
[  
K\_{(\cdot)\to n} \circ W \circ E\_{n\to (\cdot)} = \mathrm{id}*{\mathcal{H}n^\partial}  
]  
on the appropriate domain (where (K{(\cdot)\to n}) and (E*{n\to (\cdot)}) denote suitable multi-step collapse/expansion operators). Such words can be inserted into operator compositions without changing band-(n) observables.

**6.2.7 Relation to carriers and CSs**

The collapse/expansion algebra at the boundary level induces a corresponding structure on **carriers** and **Collective Spheres**:

* A band-(n) CS carries states in (\mathcal{H}\_n^\partial).
* Refining or coarsening a CS’s context (moving it to band (n+1) or (n-1)) corresponds to applying (E\_{n\to n+1}) or (K\_{n\to n-1}) to its boundary state.

Thus, the ladder algebra provides:

* a way to move CSs between context bands without losing track of how their boundary states map into each other, and
* a mechanism for relating multi-band descriptions of the same underlying present-moment structure.

This will be essential when we define the **reproduction kernel** in the next section: the kernel will encode how often and in what “patterns” states at band (n) reproduce themselves or generate new states at band (n+1), and the collapse/expansion algebra will be the substrate on which that kernel acts.

**6.2.8 Summary**

Section 6.2 has:

* Introduced **boundary state spaces** (\mathcal{H}\_n^\partial) for each band (n).
* Defined **collapse operators** (K\_{n+1\to n}) and **expansion operators** (E\_{n\to n+1}) between neighboring bands.
* Imposed a **projection/embedding relation**:
  + (K\_{n+1\to n} \circ E\_{n\to n+1} = \mathrm{id}\_{\mathcal{H}\_n^\partial}),
  + (E\_{n\to n+1} \circ K\_{n+1\to n}) is a projection on (\mathcal{H}\_{n+1}^\partial).
* Extended these maps to **multi-step collapse/expansion** between arbitrary bands.
* Noted how certain composites of (K) and (E) can be treated as **neutral ladder moves** at a given band.
* Related the ladder algebra to carriers and CSs, providing a structured way to move present-moment descriptions up and down the context ladder.

In the next subsection (6.3), we will use these operators to define the **reproduction kernel** (M) and the **memory dimension** (D\_{\mathrm{mem}}(n)), which quantify how boundary structures at each band reproduce and propagate along the ladder.

**6.3 Reproduction Kernel (M) & Memory Dimension (D\_{\mathrm{mem}}(n))**

The collapse/expansion algebra of Section 6.2 tells us **how** boundary states move up and down the ladder between neighboring bands. We now want to quantify **how much structure** is effectively *reproduced* at each band when we:

* lift boundary states to a neighboring band,
* let them propagate under the dynamics at that band, and
* collapse them back.

This is encoded in a **reproduction kernel** (or reproduction operator) acting on each band’s boundary state space, and in an associated **memory dimension** (D\_{\mathrm{mem}}(n)) that measures how many effective degrees of freedom can be stably maintained at that band.

**6.3.1 Reproduction as “up–evolve–down”**

Fix a band (n) and its boundary state space (\mathcal{H}\_n^\partial). A generic **reproduction cycle** for band-(n) boundary data consists of three abstract steps:

1. **Expand** the band-(n) state up to band (n+1) using the expansion operator:  
   [  
   \psi\_n ;\in; \mathcal{H}*n^\partial  
   ;\xrightarrow{E*{n\to n+1}};  
   \psi\_{n+1} := E\_{n\to n+1}(\psi\_n) ;\in; \mathcal{H}\_{n+1}^\partial.  
   ]
2. **Evolve** at band (n+1) under some band-specific boundary evolution operator  
   [  
   U\_{n+1} : \mathcal{H}*{n+1}^\partial \to \mathcal{H}*{n+1}^\partial,  
   ]  
   reflecting one “step” of dynamics at that band (this could arise from the tick-operator algebra, from a bandwise action, etc., but we keep it abstract here):  
   [  
   \psi\_{n+1} ;\xrightarrow{U\_{n+1}};  
   \psi'*{n+1} := U*{n+1}(\psi\_{n+1}).  
   ]
3. **Collapse** back down to band (n) using  
   [  
   K\_{n+1\to n} : \mathcal{H}\_{n+1}^\partial \to \mathcal{H}*n^\partial,  
   ]  
   to obtain a new band-(n) state  
   [  
   \psi'n := K{n+1\to n}(\psi'*{n+1}) ;\in; \mathcal{H}\_n^\partial.  
   ]

The net effect on (\psi\_n) is a single linear map on (\mathcal{H}\_n^\partial), which we now define as the **reproduction kernel** for band (n).

**6.3.2 Reproduction kernel (M\_n) on (\mathcal{H}\_n^\partial)**

**Definition 6.3.1 (Reproduction kernel at band (n)).**  
Given expansion (E\_{n\to n+1}), collapse (K\_{n+1\to n}), and bandwise evolution (U\_{n+1}), the **reproduction kernel** at band (n) is the linear operator  
[  
M\_n : \mathcal{H}*n^\partial \to \mathcal{H}n^\partial  
]  
defined by  
[  
M\_n := K{n+1\to n} \circ U*{n+1} \circ E\_{n\to n+1}.  
]

Thus, one reproduction cycle is  
[  
\psi\_n \mapsto M\_n(\psi\_n) = K\_{n+1\to n}\bigl(U\_{n+1}(E\_{n\to n+1}(\psi\_n))\bigr).  
]

Interpretation:

* (E\_{n\to n+1}) embeds (\psi\_n) into the finer/coarser band above.
* (U\_{n+1}) applies the band-(n+1) dynamics.
* (K\_{n+1\to n}) projects the result back down to band (n).

Repeated application of (M\_n) describes how band-(n) boundary data is **reproduced** (or eroded) over multiple cycles:  
[  
\psi\_n^{(k+1)} = M\_n(\psi\_n^{(k)}),\quad  
\psi\_n^{(k)} = M\_n^k(\psi\_n^{(0)}).  
]

The operator (M\_n) typically is neither unitary nor invertible; it can contract or amplify certain components, reflecting how some boundary structures are stably “remembered” across context steps and others are washed out.

**6.3.3 Spectral properties and effective degrees of freedom**

The reproduction kernel (M\_n) is a linear operator on (\mathcal{H}\_n^\partial). Its **spectral properties** describe which components of boundary structure persist under repeated reproduction cycles.

Let (\sigma(M\_n)) denote the spectrum of (M\_n), and suppose, for simplicity, that (M\_n) is diagonalizable or has a clear dominant spectral decomposition. Let ({ \lambda\_{n,i} }) be its eigenvalues (counted with algebraic multiplicity), and ({ v\_{n,i} }) the corresponding eigenvectors.

* Eigenvalues (|\lambda\_{n,i}| \approx 1) correspond to **modes that are preserved** (or only weakly attenuated) across cycles.
* Eigenvalues (|\lambda\_{n,i}| \ll 1) correspond to modes that rapidly decay and thus do not contribute to long-term memory at band (n).

To quantify the **effective number of surviving modes**, we introduce a **memory dimension** (D\_{\mathrm{mem}}(n)) for each band.

**6.3.4 Memory dimension (D\_{\mathrm{mem}}(n))**

The memory dimension is an effective fractal-like dimension of the “long-lived” subspace of (\mathcal{H}\_n^\partial) under (M\_n).

**Definition 6.3.2 (Memory dimension).**  
Let (M\_n) be the reproduction kernel at band (n). Define an “(\epsilon)-long-lived” eigenspectrum by selecting those eigenvalues with modulus above some threshold (\epsilon \in (0,1)):  
[  
\Lambda\_n(\epsilon) := { \lambda\_{n,i} \in \sigma(M\_n) ;\mid; |\lambda\_{n,i}| \ge \epsilon }.  
]

Let (N\_n(\epsilon)) be the (possibly effective) number of such eigenvalues, counting multiplicity and including an appropriate regularization if (\mathcal{H}\_n^\partial) is infinite dimensional.

The **memory dimension** (D\_{\mathrm{mem}}(n)) is defined as the exponent characterizing the scaling of (N\_n(\epsilon)) as (\epsilon \to 1^-), e.g.  
[  
N\_n(\epsilon) \sim (1-\epsilon)^{-D\_{\mathrm{mem}}(n)}  
\quad \text{as } \epsilon \to 1^-,  
]  
or, more abstractly, as the effective dimension of the subspace spanned by eigenmodes with (|\lambda\_{n,i}| \approx 1).

We do not insist on a specific analytic formula; the key points are:

* (D\_{\mathrm{mem}}(n)) is a **bandwise scalar** attached to (M\_n),
* it measures how much **independent boundary information** can be stably propagated by reproduction at band (n), and
* it provides a second dimension-like quantity, distinct from the geometric dimension (D(n)), but related to it.

**6.3.5 Hinge condition (D\_{\mathrm{mem}}(0) = 2)**

At the hinge band (n=0), we expect the memory dimension to match the hinge inner dimension, reflecting the special role of the present boundary as a **2D pivot**.

**Axiom 6.3.3 (Hinge memory dimension).**  
At the hinge band,  
[  
D\_{\mathrm{mem}}(0) = 2.  
]

This encodes the idea that:

* the hinge-level boundary **remembers** information as if it had an effective dimension 2, consistent with:
  + the IN dimension (D(0) = 2),
  + the present-plane structure (\mathcal{P}),
  + and the 2D collapse kernel that will later act on the hinge boundary.

In other words, the **number of independent long-lived modes** on the hinge boundary behaves as if they were distributed over a 2D surface—no more, no less in the idealized theory.

**6.3.6 Relationship between (D\_{\mathrm{mem}}(n)) and (D(n))**

We now state the structural relationship between the **memory dimension** (D\_{\mathrm{mem}}(n)) and the **geometric IN dimension** (D(n)).

**Axiom 6.3.4 (Dimension–memory linkage).**

For each band (n), there exists a small correction function (\delta\_{\mathrm{mem}}(n)) such that  
[  
D\_{\mathrm{mem}}(n) = D(n) + \delta\_{\mathrm{mem}}(n),  
]  
with the following properties:

1. **Hinge match**  
   [  
   \delta\_{\mathrm{mem}}(0) = 0,  
   ]  
   so (D\_{\mathrm{mem}}(0) = D(0) = 2).
2. **Bounded deviation**  
   [  
   |\delta\_{\mathrm{mem}}(n)| \le C  
   ]  
   for some constant (C) and all (n).
3. **Qualitative alignment**
   * For bands with larger geometric dimension (D(n)) (inner, more volume-like regimes), (D\_{\mathrm{mem}}(n)) is also larger, reflecting greater capacity to store and reproduce boundary information.
   * For bands with smaller geometric dimension (D(n)) (outer, more filamentary regimes), (D\_{\mathrm{mem}}(n)) is smaller, reflecting reduced memory capacity.

In many idealized models, one sets (\delta\_{\mathrm{mem}}(n) \equiv 0) and simply identifies  
[  
D\_{\mathrm{mem}}(n) = D(n),  
]  
but the formalism allows for small corrections.

**6.3.7 Compatibility with the radial profile**

The radial picture (D(r)) can be imported into the bandwise memory description via the mapping (n\mapsto r(n)):

* Define a radial memory dimension (D\_{\mathrm{mem}}(r)) and require  
  [  
  D\_{\mathrm{mem}}(n) = D\_{\mathrm{mem}}(r(n)).  
  ]
* Align this with the radial present exponent and IN dimension as:  
  [  
  D\_{\mathrm{mem}}(r) \approx D(r) \approx d\_{\mathrm{PMS}}(r),  
  ]  
  with equalities at the hinge and small bounded deviations away from it.

Thus, **the same hinge** appears in:

* the geometric IN dimension (D(r)),
* the present exponent (d\_{\mathrm{PMS}}(r)), and
* the memory dimension (D\_{\mathrm{mem}}(r)).

**6.3.8 Summary**

Section 6.3 has:

* Defined the **reproduction kernel** (M\_n = K\_{n+1\to n} \circ U\_{n+1} \circ E\_{n\to n+1}) on each band’s boundary state space (\mathcal{H}\_n^\partial).
* Interpreted (M\_n) as the net “up–evolve–down” map that describes how band-(n) boundary states reproduce across context steps.
* Introduced the **memory dimension** (D\_{\mathrm{mem}}(n)) as an effective dimension of the long-lived subspace of (\mathcal{H}*n^\partial) under repeated application of (M\_n), based on the distribution of eigenvalues (|\lambda*{n,i}|) near 1.
* Imposed the hinge condition (D\_{\mathrm{mem}}(0) = 2), aligning memory capacity at the hinge with the hinge inner dimension and present-plane structure.
* Stated a general linkage between (D\_{\mathrm{mem}}(n)) and the geometric dimension (D(n)), with small bounded deviations, and extended this to the radial picture.

In the next subsection (6.4), we will make the **compatibility with the radial profile (D(r))** explicit, ensuring that the discrete ladder curve (D(n)) and the memory dimension (D\_{\mathrm{mem}}(n)) fit consistently with the continuous description and the hinge-centered behavior developed in Part V.

**6.4 Compatibility with the Radial Profile (D(r))**

The discrete ladder ({D(n), g(D(n)), D\_{\mathrm{mem}}(n)}) was built as a banded version of the continuous radial picture ({D(r), g(D(r)), D\_{\mathrm{mem}}(r)}) from Part V. In this section we state the **compatibility conditions** between these two descriptions:

* how band indices (n) map to radial positions (r),
* how (D(n)) approximates (D(r)),
* and how the discrete memory dimension (D\_{\mathrm{mem}}(n)) aligns with the radial present exponent and memory dimension.

The goal is to ensure that the discrete ladder is a **faithful sampling** of the radial structure, especially near the hinge.

**6.4.1 Sampling map (n \mapsto r(n))**

We start by making the mapping from the discrete index (n) to the radial parameter (r) explicit.

**Definition 6.4.1 (Sampling map).**  
A **sampling map** is a function  
[  
r : \mathbb{Z} \to \mathbb{R},\quad n \mapsto r(n),  
]  
such that:

1. **Hinge alignment**  
   [  
   r(0) = 0.  
   ]
2. **Monotonicity**  
   [  
   n\_1 < n\_2 ;\Rightarrow; r(n\_1) < r(n\_2),  
   ]  
   so that inner bands ((n<0)) map to inward radial positions ((r<0)) and outer bands ((n>0)) map to outward positions ((r>0)).
3. **Scale regularity**  
   There exists a band spacing (\Delta r > 0) and a bounded correction (\epsilon\_n) such that  
   [  
   r(n) = n \Delta r + \epsilon\_n,  
   ]  
   with (|\epsilon\_n|) bounded uniformly in (n). In the simplest case, (r(n)=n\Delta r).

Using this mapping, we define the **sampled profiles**:  
[  
D(n) := D(r(n)), \quad  
D\_{\mathrm{mem}}(n) := D\_{\mathrm{mem}}(r(n)),  
]  
and similarly for other radial functions.

**6.4.2 Bandwise dimension as sampled radial dimension**

The bandwise dimension profile ({D(n)}) is required to approximate the radial profile (D(r)) at the sampling points.

**Axiom 6.4.2 (Dimension sampling consistency).**

For all (n \in \mathbb{Z}),  
[  
D(n) = D(r(n)),  
]  
with:

1. **Hinge match**  
   [  
   D(0) = D(r(0)) = D(0)\_{\text{radial}} = 2.  
   ]
2. **Monotone behavior in the large**  
   For sufficiently large (|n|), the sign and approximate magnitude of (D(n)-2) matches that of (D(r(n))-2), with inward bands ((n<0)) having (D(n)>2) and outward bands ((n>0)) having (D(n)<2), consistent with the asymptotics of (D(r)).
3. **Local Lipschitz compatibility**  
   There exists a constant (L) such that for any pair of bands (n\_1,n\_2),  
   [  
   |D(n\_2) - D(n\_1)| \le L, |r(n\_2) - r(n\_1)|,  
   ]  
   induced from the Lipschitz behavior of (D(r)) on the radial domain.

Thus, the discrete ladder does not introduce spurious oscillations or discontinuities in dimension; it is a coarse sampling of a smooth radial curve.

**6.4.3 Bandwise memory dimension vs radial memory dimension**

The memory dimension (D\_{\mathrm{mem}}(n)) defined via the reproduction kernel in Section 6.3 must be compatible with a radial memory profile (D\_{\mathrm{mem}}(r)) that incorporates the same notions as in the continuous picture.

We require:

**Axiom 6.4.3 (Memory sampling consistency).**

For all (n \in \mathbb{Z}),  
[  
D\_{\mathrm{mem}}(n) = D\_{\mathrm{mem}}(r(n)),  
]  
with:

1. **Hinge condition**  
   [  
   D\_{\mathrm{mem}}(0) = D\_{\mathrm{mem}}(r(0)) = 2.  
   ]
2. **Qualitative alignment with (D(r)) and (d\_{\mathrm{PMS}}(r))**  
   For all (r),  
   [  
   D\_{\mathrm{mem}}(r) \approx D(r) \approx d\_{\mathrm{PMS}}(r),  
   ]  
   with exact equality at (r=0) and small bounded deviations elsewhere. Consequently,  
   [  
   D\_{\mathrm{mem}}(n) \approx D(n) \approx d\_{\mathrm{PMS}}(r(n)).  
   ]

This ensures that the **geometric dimension**, the **present exponent**, and the **memory dimension** all share the same hinge and vary coherently with radial context, both in the continuous and discrete descriptions.

**6.4.4 Ladder version of mirror behavior**

The approximate mirror symmetry of the radial profile (D(r)) and gate function (g(D(r))) under (r \to -r) (near the hinge) translates into a ladder version under (n \to \bar{n}) (often (\bar{n}=-n)).

**Axiom 6.4.4 (Bandwise mirror compatibility).**

There exists an involutive map (\bar{n} : \mathbb{Z} \to \mathbb{Z}) (i.e. (\bar{\bar{n}} = n)) with (\bar{0} = 0) such that:

1. For bands close to the hinge,  
   [  
   r(\bar{n}) \approx -r(n),  
   ]  
   so that  
   [  
   D(\bar{n}) = D(r(\bar{n})) \approx D(-r(n)),\quad  
   g(D(\bar{n})) \approx g(D(-r(n))).  
   ]
2. In a perfectly symmetric model, one can choose (\bar{n} = -n) and obtain exact relations, e.g.  
   [  
   D(-n) + D(n) = 4  
   ]  
   and  
   [  
   g(D(-n)) = g(D(n)).  
   ]
3. The memory dimension respects the same mirror pairing:  
   [  
   D\_{\mathrm{mem}}(\bar{n}) \approx D\_{\mathrm{mem}}(n),  
   ]  
   reflecting that inner and outer bands paired in this way have comparable memory capacity, viewed through the hinge.

This ladder mirror structure is the discrete counterpart of the radial mirror symmetry, with the hinge band (n=0) acting as the fixed point.

**6.4.5 Action and kernel consistency across radial and discrete descriptions**

The **master action** and **reproduction kernel** can be expressed either in radial or discrete form:

* Radial form:  
  [  
  S\_{\text{cont}} \sim \int g(D(r)), \mathcal{L}(r), \mathrm{d}r,  
  ]
* Discrete ladder form:  
  [  
  S\_{\text{disc}} \sim \sum\_{n} g(D(n)), \Delta S\_n.  
  ]

Compatibility requires that the discrete version converges to the continuous one in an appropriate limit, e.g. as (\Delta r \to 0) and band spacing becomes fine.

**Axiom 6.4.5 (Action-consistency).**

Let (\Delta r) be the effective radial spacing between bands, and suppose (\mathcal{L}(r)) is a bandwise Lagrangian density. Then for sufficiently fine ladder spacing:

[  
\sum\_{n} g(D(n)), \mathcal{L}(r(n)), \Delta r  
;\approx;  
\int g(D(r)), \mathcal{L}(r), \mathrm{d}r.  
]

Similarly, the discrete reproduction kernels (M\_n) and the radial memory operator (M(r)) (if defined) must approximate each other under the sampling map:

[  
M\_n ;\approx; M(r(n)),  
]  
in the sense that:

* their dominant eigenvalues and eigenvectors correspond under the mapping, and
* the bandwise memory dimensions (D\_{\mathrm{mem}}(n)) approximate the radial memory dimension (D\_{\mathrm{mem}}(r(n))), as already required by Axiom 6.4.3.

**6.4.6 Hinge as common fixed point**

In both the radial and discrete descriptions, the hinge is the unique common pivot:

* Radial hinge: (r=0) with  
  [  
  D(0) = 2,\quad g(D(0)) = 1,\quad D\_{\mathrm{mem}}(0) = 2,\quad d\_{\mathrm{PMS}}(0) = 2.  
  ]
* Ladder hinge: (n=0) with  
  [  
  D(0) = 2,\quad g(D(0)) = 1,\quad D\_{\mathrm{mem}}(0) = 2.  
  ]

Compatibility requires that:

* the sampling map preserves this fixed point ((r(0)=0)),
* the hinge serves simultaneously as:
  + the **geometric pivot** (2D IN boundary),
  + the **memory pivot** (2D long-lived modes),
  + and the **present pivot** (2D present-plane structure).

Thus, all three notions of “2D pivot” coincide in both radial and discrete frameworks.

**6.4.7 Summary**

Section 6.4 has:

* Introduced a **sampling map** (n \mapsto r(n)) connecting the discrete ladder to the continuous radial parameter.
* Required that the bandwise dimension profile (D(n)) and memory dimension (D\_{\mathrm{mem}}(n)) be sampled from their radial counterparts (D(r)) and (D\_{\mathrm{mem}}(r)), with hinge alignment and consistent monotonic behavior.
* Imposed a **bandwise mirror compatibility** condition that mirrors the radial symmetry (r \to -r) near the hinge, encoding inner/outer reciprocity in the discrete ladder.
* Stated **action-consistency** between radial integrals and discrete band sums, and aligned reproduction kernels and memory dimensions across radial and discrete descriptions.
* Emphasized that the hinge is a **common fixed point** of the geometric, memory, and present exponent structures in both pictures.

With this, the discrete ladder and the radial structure are fully aligned. In the next subsection (6.5), we will specify a concrete and commonly used shape for the discrete dimension curve (D(n)), typically a **logistic-type profile** that interpolates between inner (volume-like) and outer (filament-like) regimes while satisfying all hinge and reciprocity conditions.

**6.5 Logistic Dimension Curve (D(n))**

We now specify a concrete, but still structural, form for the **discrete dimension profile** (D(n)) on the context ladder. The aim is to capture:

* inner bands ((n \ll 0)) with **volume-like** dimension ((D>2)),
* outer bands ((n \gg 0)) with **filament-like** dimension ((D<2)),
* a **hinge band** (n=0) with (D(0)=2),
* and a smooth, monotone interpolation between these regimes.

A convenient choice is a **logistic-type curve**, which approximates the qualitative shape we need and is compatible with the radial profile (D(r)).

**6.5.1 Logistic-type ansatz**

We choose a parameterized form for (D(n)) that:

* tends to a value (D\_{\text{in}}>2) as (n \to -\infty),
* tends to a value (D\_{\text{out}}<2) as (n \to +\infty),
* and passes through (D(0)=2).

A canonical choice is:

**Definition 6.5.1 (Logistic dimension curve).**  
Let (k>0) be a “steepness” parameter and set the hinge band to (n\_0=0). Define  
[  
D(n) := D\_{\text{out}} + \frac{D\_{\text{in}} - D\_{\text{out}}}{1 + e^{k n}},  
]  
with parameters satisfying  
[  
1 \le D\_{\text{out}} < 2 < D\_{\text{in}} \le 3.  
]

This gives:

* As (n \to -\infty): (e^{k n} \to 0) and  
  [  
  D(n) \to D\_{\text{in}}.  
  ]
* As (n \to +\infty): (e^{k n} \to \infty) and  
  [  
  D(n) \to D\_{\text{out}}.  
  ]

The hinge condition (D(0)=2) imposes  
[  
2 = D\_{\text{out}} + \frac{D\_{\text{in}} - D\_{\text{out}}}{1 + e^{0}}  
= D\_{\text{out}} + \frac{D\_{\text{in}} - D\_{\text{out}}}{2},  
]  
so that  
[  
D\_{\text{in}} + D\_{\text{out}} = 4.  
]

Thus, in this particular logistic ansatz,  
[  
D\_{\text{out}} = 4 - D\_{\text{in}},  
]  
and the inner/outer limiting dimensions are related by a simple mirror symmetry around 2.

This explicit parameterization is not mandatory, but it is prototypical of the class of curves we consider: smooth, sigmoidal, and symmetric (or nearly so) about the hinge.

**6.5.2 Qualitative properties**

Regardless of exact parameter choices, any logistic-type (D(n)) used in the theory is required to satisfy:

1. **Hinge value**  
   [  
   D(0) = 2.  
   ]
2. **Inner/outer asymptotics**  
   [  
   \lim\_{n\to -\infty} D(n) = D\_{\text{in}} \in (2,3],\quad  
   \lim\_{n\to +\infty} D(n) = D\_{\text{out}} \in [1,2),  
   ]  
   with (D\_{\text{in}} > 2 > D\_{\text{out}}).
3. **Monotonicity**
   * (D(n)) is strictly decreasing in (n): as we move from inner to outer bands, the effective inner dimension drops.
4. **Smoothness (discrete sense)**
   * The discrete derivative  
     [  
     \Delta D(n) := D(n+1) - D(n)  
     ]  
     is bounded and varies slowly with (n), i.e. there is a constant (L>0) such that  
     [  
     |\Delta D(n+1) - \Delta D(n)| \le L  
     ]  
     for all (n). In logistic models, (\Delta D(n)) itself has a smooth, peaked shape around the hinge.

These properties ensure that the ladder transitions smoothly from inner to outer regimes through a well-defined 2D hinge layer.

**6.5.3 Mirror symmetry and band pairing**

The logistic form above naturally supports a **mirror symmetry** under (n \to -n) when (D\_{\text{in}} + D\_{\text{out}} = 4). In that case,  
[  
D(-n) + D(n) = 4 \quad \text{for all } n,  
]  
or equivalently  
[  
D(-n) - 2 = -(D(n) - 2).  
]

This is a discrete version of the radial mirror behavior (D(-r)+D(r)=4) discussed earlier. It implies:

* Each inner band (n<0) has an outer partner band (+n) such that their distances from the hinge dimension 2 are equal and opposite.
* Hinge band (n=0) is self-paired with (D(0)=2).

In more general models, the symmetry may only be approximate, or the pairing might be defined by an involution (\bar{n}) different from (-n), but the logistic curve serves as a clean symmetric reference case.

**6.5.4 Relation to the memory dimension (D\_{\mathrm{mem}}(n))**

As in Section 6.3, the memory dimension (D\_{\mathrm{mem}}(n)) characterizes the number of long-lived modes under the reproduction kernel (M\_n). For a logistic (D(n)), we typically require that (D\_{\mathrm{mem}}(n)) **follows the same logistic shape** up to small corrections.

Concretely, we can write  
[  
D\_{\mathrm{mem}}(n) = D(n) + \delta\_{\mathrm{mem}}(n),  
]  
with

* (\delta\_{\mathrm{mem}}(0) = 0),
* (|\delta\_{\mathrm{mem}}(n)|) bounded and small compared to (|D(n)-2|),
* and, in symmetric models,  
  [  
  \delta\_{\mathrm{mem}}(-n) \approx \delta\_{\mathrm{mem}}(n).  
  ]

Thus, bands with higher geometric dimension (D(n)) (inner) have correspondingly higher memory dimension, and bands with lower dimension (D(n)) (outer) have lower memory dimension, in a way that mirrors the logistic profile.

**6.5.5 Steepness and localization around the hinge**

The parameter (k) in the logistic form controls **how sharply** the dimension profile transitions from inner to outer regimes:

* **Small (k)**: a **gentle** transition, with a broad range of bands near the hinge where (D(n)) remains close to 2.
* **Large (k)**: a **sharp** transition, where only a small number of bands (e.g. (n=0,\pm 1)) have (D(n)) near 2, and the dimension rapidly approaches its inner/outer asymptotes.

In structural terms:

* A gentler transition (small (k)) corresponds to a **thicker hinge zone**, where many bands experience near-2D behavior.
* A sharper transition (large (k)) corresponds to a **thin hinge zone**, with a tightly localized 2D boundary layer.

The formalism accommodates both cases; the choice of (k) affects how tightly various hinge-related phenomena (collapse kernels, area-law behavior, inverse-square scaling) are localized around (n=0).

**6.5.6 Gate weights (g\_n) induced by the logistic profile (preview)**

The logistic curve for (D(n)) induces bandwise gate weights  
[  
g\_n := g(D(n)),  
]  
which inherit the smooth, hinge-centered shape of (D(n)):

* (g\_0 = g(D(0)) = g(2) = 1),
* (g\_n) varies slowly in (n),
* and in symmetric models (g\_{-n} \approx g\_n), reflecting inward/outward reciprocity.

In Section 6.6 we will use this (D(n)) to formulate the **hinge area-law boundary** and interpret (D(0)=2) in terms of an emergent 2D “surface” supporting inverse-square behavior and area-proportional measures.

**6.5.7 Summary**

Section 6.5 has:

* Chosen a **logistic-type ansatz** for the discrete dimension curve (D(n)) that:
  + interpolates smoothly between inner dimensions (D\_{\text{in}}>2) and outer dimensions (D\_{\text{out}}<2),
  + passes through the hinge value (D(0)=2), and
  + can satisfy a mirror symmetry (D(-n)+D(n)=4) in symmetric cases.
* Stated the qualitative properties of (D(n)): monotone decreasing, smooth in a discrete sense, and consistent with the radial profile.
* Related (D(n)) to the memory dimension (D\_{\mathrm{mem}}(n)), which typically follows the same logistic shape up to small corrections.
* Highlighted the role of the steepness parameter (k) in controlling the thickness of the hinge zone around (n=0).
* Prepared for Section 6.6, where we will combine this logistic (D(n)) with the pivot function (g(D(n))) to interpret the hinge band as an **area-law boundary** that naturally supports inverse-square fields and plays a central role in the emergent gravitational and gauge structures.

**6.6 Pivot Function (g(D(n))) & Hinge as Area-Law Boundary**

We now combine the discrete dimension curve (D(n)) with the pivot function (g(D)) to define **bandwise pivot weights** (g\_n) and interpret the hinge band (n=0) as an **area-law boundary**:

* inner bands (n \ll 0): more volume-like,
* hinge band (n=0): effectively 2-dimensional “surface” (area-law),
* outer bands (n \gg 0): more filament-like.

This is the discrete ladder counterpart of the radial hinge picture from Part V, and it is the structural basis for inverse-square behavior and surface-based counting used later in the field and gravity sectors.

**6.6.1 Bandwise pivot weights**

Given the bandwise dimension profile (D(n)), we define pivot weights on each band by sampling the pivot function (g(D)):

**Definition 6.6.1 (Bandwise pivot weights).**  
For each band (n \in \mathbb{Z}), define  
[  
g\_n := g(D(n)).  
]

Properties:

1. **Hinge normalization**  
   [  
   g\_0 = g(D(0)) = g(2) = 1.  
   ]
2. **Smooth variation across bands**  
   Because (D(n)) varies smoothly (in the discrete sense) and (g(D)) is continuous and Lipschitz near (D=2), the sequence ({g\_n}) varies smoothly in (n):  
   [  
   |g\_{n+1} - g\_n| \le L, |D(n+1) - D(n)|  
   ]  
   for some constant (L>0).
3. **Mirror behavior (in symmetric models)**  
   In models with logistic (D(n)) satisfying (D(-n)+D(n)=4) and a symmetric pivot function, we have approximately  
   [  
   g\_{-n} \approx g\_n,  
   ]  
   so inner and outer bands mirrored around the hinge carry comparable gate weights.

The (g\_n) will:

* scale band contributions in the master action, and
* modulate cross-band couplings, with (g\_0=1) as the reference.

**6.6.2 Boundary measure and effective “area” at the hinge**

To speak of an **area-law boundary**, we need a notion of “measure” on each boundary graph (\mathcal{G}\_n).

For each band (n), let (\mathcal{G}\_n=(V\_n,E\_n)) be the boundary graph (Section 6.1), and let (\mathsf{dist}\_n(\cdot,\cdot)) be a graph-distance (shortest-path length) on (V\_n).

**Definition 6.6.2 (Bandwise boundary balls and measures).**

Fix a reference node (v\_{n,0} \in V\_n). For integer (R\ge 1), define the **boundary ball** of graph radius (R) at band (n) as  
[  
B\_n(R) := { v \in V\_n \mid \mathsf{dist}*n(v, v*{n,0}) \le R }.  
]

Let (\mu\_n) be a non-negative measure on (V\_n) (e.g. counting measure or a normalized version). Then the **boundary measure at scale (R)** is  
[  
A\_n(R) := \mu\_n(B\_n(R)).  
]

We require that, for sufficiently large (R) in the asymptotic regime of band (n),  
[  
A\_n(R) \propto R^{D(n)},  
]  
in the sense that the logarithmic growth rate of (A\_n(R)) with (R) matches the effective dimension (D(n)).

At the hinge, (D(0)=2), so  
[  
A\_0(R) \propto R^2,  
]  
and we interpret the hinge boundary graph (\mathcal{G}\_0) as **effectively 2-dimensional**: its boundary measure grows like an area.

This is what we mean when we say that band (n=0) is an **area-law boundary**.

**6.6.3 Hinge as unique area-law pivot between volume and filament**

Given the logistic-type (D(n)):

* inner bands (n\ll0): (D(n) \to D\_{\text{in}} > 2) (volume-like),
* outer bands (n\gg0): (D(n) \to D\_{\text{out}} < 2) (filament-like),
* hinge band (n=0): (D(0)=2) (area-like).

The hinge is the **unique band** where:

* boundary measure scales as (A\_0(R)\propto R^2), and
* the pivot weight is unscaled, (g\_0=1).

All other bands satisfy:

* **Inner bands ((n<0))**: (D(n)>2), so (A\_n(R) \propto R^{D(n)}) grows faster than (R^2); boundaries look “thicker” than surfaces (volume-like behavior).
* **Outer bands ((n>0))**: (D(n)<2), so (A\_n(R) \propto R^{D(n)}) grows more slowly than (R^2); boundaries look “thinner” than surfaces (filament-like behavior).

Thus the hinge band (n=0) sits exactly between:

* dense, volume-like inner structure and
* sparse, filament-like outer structure,

and it is the **only** band with **pure area-law** scaling. This is why it is singled out as the **pivot band** in the ladder.

**6.6.4 Band contributions in the master action**

In the discrete master action (developed in Part VII), band contributions have the schematic form  
[  
S\_{\text{disc}} \sim \sum\_{n} g\_n, \Delta S\_n,  
]  
where (\Delta S\_n) is the band-(n) contribution derived from boundary states on (\mathcal{G}\_n) and their dynamics.

Because:

* (g\_0=1),
* (D(0)=2) (area-law boundary),
* and inner/outer bands deviate from 2 in opposite directions,

the hinge band often plays a **distinguished role** in the action:

* terms involving band (n=0) are **un-weighted** by (g\_n),
* many “surface” or flux-like quantities (e.g. those leading to inverse-square laws) are naturally tied to this area-law boundary,
* contributions from bands far from the hinge can be suppressed or enhanced depending on the precise shape of ({g\_n}) and ({D(n)}), but they are structurally distinct from the hinge.

This is the discrete analogue of treating a 2D surface in a radial continuum as the primary carrier of flux or entropy, with interior and exterior contributions folded into the weighting.

**6.6.5 Hinge band and inverse-square behavior (structural preview)**

Although the detailed derivation of inverse-square laws is postponed to later parts (especially when collapse kernels on (\mathcal{G}\_0) and null-cone structures are introduced), we can already state the **structural mechanism**:

1. At band (n=0), boundary measure scales as (A\_0(R)\propto R^2).
2. If some conserved quantity (e.g. a flux or “charge”) is distributed uniformly over this hinge boundary, then the **density per unit boundary measure** scales as  
   [  
   \text{density}(R) \propto \frac{1}{A\_0(R)} \propto \frac{1}{R^2}.  
   ]
3. The hinge band is the natural place to define such distributions, because:
   * it is the unique area-law layer, and
   * gate weights are normalized, so no extra scaling is applied.

When this idea is combined with:

* the collapse kernels acting on (\mathcal{G}\_0) (later in the theory), and
* the present-act dynamics,

we obtain **inverse-square fields** as emergent structures tied to the hinge. The present section provides the combinatorial and dimensional backdrop for that derivation.

**6.6.6 Summary**

Section 6.6 has:

* Defined **bandwise pivot weights** (g\_n = g(D(n))) with hinge normalization (g\_0=1).
* Introduced a notion of **boundary measure** (A\_n(R)) on each boundary graph (\mathcal{G}\_n), with asymptotic scaling (A\_n(R)\propto R^{D(n)}).
* Identified the hinge band (n=0) as the **unique area-law boundary** where (D(0)=2) and (A\_0(R)\propto R^2), sitting between volume-like inner bands and filament-like outer bands.
* Explained how this hinge band plays a special role in the master action and why it naturally supports **inverse-square behavior** when flux-like quantities are distributed over it.

With the discrete ladder, reproduction kernel, and hinge area-law structure in place, we are ready in Part VII to define the **discrete master action** on the ladder, take its continuum embedding in context-time, and develop the field-theoretic and quantum structures that arise from this hinge-weighted context dynamics.

**7. Master Action, Quantization & Renormalization**

**7.1 Discrete Master Action on the Ladder**

We now introduce the **discrete master action** that governs dynamics on the context ladder. The idea is:

* treat the **ladder index** (n) as labeling bands,
* introduce a separate **context-time** index (or step) labeling updates of the ladder state, and
* define an action functional whose variation yields the ladder–evolution equations, with each band’s contribution weighted by its pivot factor (g(D(n))).

This section defines the action at the discrete level; the continuum embedding and quantization are handled in later subsections.

**7.1.1 Context-time and ladder state**

We introduce a discrete **context-time index**  
[  
m \in \mathbb{Z},  
]  
which labels successive “configuration steps” of the ladder (not to be confused with the tick index (k) used for present-moment evolution in Sections 2–3).

At each context-time step (m), we have boundary states on all bands:

* For band (n), a boundary state  
  [  
  \psi\_n(m) \in \mathcal{H}\_n^\partial,  
  ]  
  where (\mathcal{H}\_n^\partial) is the boundary state space associated with the boundary graph (\mathcal{G}\_n) (Section 6.2).

We collect all bandwise boundary states into a single **ladder state**  
[  
\Psi(m) := {\psi\_n(m)}*{n\in\mathbb{Z}},  
]  
which we may regard as an element of the direct sum  
[  
\mathcal{H}*\text{ladder}^\partial := \bigoplus\_{n\in\mathbb{Z}} \mathcal{H}\_n^\partial.  
]

Thus, the ladder dynamics is described by the sequence  
[  
\dots, \Psi(m-1), \Psi(m), \Psi(m+1), \dots  
]  
in context-time.

**7.1.2 Local band Lagrangians**

We define **local Lagrangian functionals** for each band, which depend on:

* the band’s state at adjacent context-time steps, (\psi\_n(m)) and (\psi\_n(m+1)),
* possibly neighboring bands’ states via the collapse/expansion maps (K\_{n\pm1\to n}), (E\_{n\to n\pm1}), and
* bandwise parameters (e.g. (D(n)), (g\_n)).

**Definition 7.1.1 (Bandwise Lagrangian).**  
For each band (n), a **band Lagrangian** is a real-valued functional  
[  
\mathcal{L}*n\bigl(\psi\_n(m+1), \psi\_n(m); {\psi*{n'}(m+1),\psi\_{n'}(m)}\_{n'=\text{neighbors of }n}\bigr),  
]  
which is:

1. **Local in context-time:** depends only on (\psi\_n(m)), (\psi\_n(m+1)) and (optionally) on neighboring bands’ states at the same two time steps; no dependence on more distant context-time steps.
2. **Band-local with controlled cross-band couplings:** any dependence on bands (n'\neq n) enters only via prescribed collapse/expansion combinations such as  
   [  
   K\_{n+1\to n}\psi\_{n+1}(m), \quad E\_{n-1\to n}\psi\_{n-1}(m),  
   ]  
   ensuring compatibility with the ladder algebra of Section 6.2.
3. **Gauge- and ladder-compatible:** respects the neutral moves (both tick-level and ladder-level) and the symmetry structure from earlier sections, so that adding neutral sequences does not change the action.

We do not fix a specific analytic form for (\mathcal{L}\_n); it can be quadratic, higher-order, or more general, depending on what effective field content we want to encode. The key requirement is that it is **well-defined** on (\mathcal{H}\_n^\partial) and compatible with the ladder structure.

**7.1.3 Band-weighted discrete master action**

We now assemble the band Lagrangians into a single discrete action, with each band contribution weighted by its pivot factor (g\_n = g(D(n))).

**Definition 7.1.2 (Discrete master action).**  
Given a ladder state history ({\Psi(m)}*{m=m\_0}^{m\_1}), the* ***discrete master action*** *is  
[  
S*{\text{disc}}[\Psi] := \sum\_{m=m\_0}^{m\_1-1} \sum\_{n\in\mathbb{Z}} g\_n,  
\mathcal{L}*n\bigl(\psi\_n(m+1), \psi\_n(m); {\psi*{n'}(m+1),\psi\_{n'}(m)}\_{n'=\text{neighbors of }n}\bigr).  
]

Key features:

* The **inner sum over (n)** adds contributions from all bands at a given context-time step (m).
* The **outer sum over (m)** accumulates contributions along context-time.
* The **pivot weights** (g\_n = g(D(n))) modulate the relative contribution of each band, with the hinge band (n=0) unweighted (since (g\_0 = 1)).

For notational simplicity, we can write  
[  
S\_{\text{disc}}[\Psi] = \sum\_{m} \mathcal{L}*{\text{tot}}(\Psi(m+1),\Psi(m)),  
]  
with  
[  
\mathcal{L}*{\text{tot}}(\Psi(m+1),\Psi(m))  
:= \sum\_{n} g\_n, \mathcal{L}\_n\bigl(\psi\_n(m+1),\psi\_n(m); \ldots \bigr).  
]

**7.1.4 Neutral-move invariance**

Because the flip algebra and ladder algebra both admit **neutral moves** (Section 2.4 and Section 6.2), the master action must be invariant under inserting such moves into the history, as long as they are properly localized and do not alter the bandwise states at the endpoints.

Formally:

**Axiom 7.1.3 (Neutral-move invariance).**

Let (\Psi(m)) be a ladder state history. Consider modifying this history by inserting, between (\Psi(m)) and (\Psi(m+1)), a finite sequence of:

* tick-level neutral words (that act as identity on the carriers in each band), and/or
* ladder-level neutral composites of collapse/expansion that act as identity on the bandwise boundary states.

Then:

* the modified history has the same bandwise boundary states (\psi\_n(m)), (\psi\_n(m+1)) at those time steps as the original, and
* the master action satisfies  
  [  
  S\_{\text{disc}}[\Psi\_{\text{modified}}] = S\_{\text{disc}}[\Psi].  
  ]

This ensures that (S\_{\text{disc}}) depends only on the **equivalence class** of histories under neutral moves, consistent with the earlier discussion that physical invariants are attached to flip-count and ladder-equivalence classes rather than specific micro-sequences.

**7.1.5 Hinge band’s distinguished role**

Because:

* (D(0)=2),
* (g\_0 = g(D(0)) = 1),
* and the hinge boundary graph (\mathcal{G}\_0) satisfies an area law,

the hinge band contributes  
[  
\sum\_{m} \mathcal{L}\_0(\psi\_0(m+1),\psi\_0(m); \ldots)  
]  
to the action **without any additional weighting**. All other bands are scaled relative to this hinge contribution:

* inner bands ((n<0)) with (D(n)>2) and (g\_n\neq 1) contribute terms that can be interpreted as **interior corrections** or **volume-like refinements** of hinge-level dynamics,
* outer bands ((n>0)) with (D(n)<2) and (g\_n\neq 1) contribute **ambient corrections** or **filament-like extensions**.

Structurally, the hinge band is the “baseline” around which the rest of the ladder is organized in the action.

**7.1.6 Equations of motion (preview)**

The discrete equations of motion for the ladder state follow from **stationary action**:

* Vary the action with respect to (\psi\_n(m)), subject to appropriate boundary conditions (fixed initial and final ladder states, or boundary conditions at infinity).
* Set the first variation to zero:  
  [  
  \delta S\_{\text{disc}}[\Psi] = 0.  
  ]

This yields a set of coupled discrete Euler–Lagrange equations  
[  
\mathcal{E}*n\bigl(\psi\_n(m+1),\psi\_n(m),\psi\_n(m-1); {\psi*{n'}(\cdot)}\bigr) = 0  
]  
for each band (n) and each context-time step (m).

We do not yet write these equations explicitly; they will be discussed in Section 7.3 after we introduce the continuum embedding and Hamiltonian structure in Section 7.2.

**7.1.7 Summary**

Section 7.1 has:

* Introduced a discrete **context-time** index (m) and a combined ladder state (\Psi(m) = {\psi\_n(m)}\_n).
* Defined **bandwise Lagrangians** (\mathcal{L}\_n) that depend on band states at adjacent context-time steps and, optionally, on neighboring bands via the collapse/expansion maps.
* Constructed the **discrete master action**  
  [  
  S\_{\text{disc}}[\Psi] = \sum\_m \sum\_n g\_n, \mathcal{L}\_n(\cdots),  
  ]  
  with pivot weights (g\_n = g(D(n))) and hinge normalization (g\_0=1).
* Imposed **neutral-move invariance**, ensuring that the action depends only on physically relevant equivalence classes of histories.
* Highlighted the hinge band (n=0) as a distinguished, unweighted contribution with area-law boundary behavior.

In the next subsection (7.2), we will take the **continuum limit in context-time**, introduce a **context-time parameter** and a **Lagrangian density**, and define the **division-by-zero operator** that encodes the special role of the hinge band in the continuum action.

**7.2 Continuum Action in Context-Time & Division-by-Zero Operator**

Section 7.1 defined a **discrete** master action on the context ladder, indexed by a discrete context-time step (m). In this section we:

1. Take a **continuum limit in context-time**, obtaining an integral over a continuous parameter.
2. Introduce a formal **division-by-zero operator** that implements:
   * projection of the context-time integral onto the **hinge band** (where (D(0)=2)), and
   * “thickening” of the hinge into a higher-dimensional domain (e.g. a 4-dimensional manifold) on which a standard field-theory-like action can be written.

The division-by-zero operator is a **purely structural device**: it is not literal numeric (1/0), but a compact way of describing “collapse to the hinge and promotion to a higher-dimensional action.”

**7.2.1 Continuum limit in context-time**

We start from the discrete master action of Section 7.1:  
[  
S\_{\text{disc}}[\Psi]  
= \sum\_{m=m\_0}^{m\_1-1}  
\sum\_{n\in\mathbb{Z}}  
g\_n, \mathcal{L}\_n\bigl(\psi\_n(m+1),\psi\_n(m); \dots\bigr),  
]  
where:

* (m) labels **context-time steps**,
* (n) labels **bands** on the context ladder,
* (\psi\_n(m) \in \mathcal{H}\_n^\partial) are bandwise boundary states,
* (g\_n = g(D(n))) are pivot weights.

**(a) Introducing a continuous context-time parameter**

Let (\sigma) be a continuous context-time parameter, and let the discrete index (m) sample (\sigma) with spacing (\Delta\sigma):  
[  
\sigma\_m := m,\Delta\sigma.  
]

We assume that, for sufficiently small (\Delta\sigma), the bandwise states are values of smooth functions  
[  
\psi\_n(m) \approx \psi\_n(\sigma\_m),  
]  
so that finite differences can be approximated by derivatives:  
[  
\frac{\psi\_n(m+1) - \psi\_n(m)}{\Delta\sigma}  
;\approx; \partial\_\sigma \psi\_n(\sigma)\big|\_{\sigma=\sigma\_m}.   
]

**(b) Bandwise continuum Lagrangian density**

We now introduce a **bandwise Lagrangian density** (\mathcal{L}*n^{\text{cont}}) for each band (n), depending on (\psi\_n(\sigma)) and its context-time derivative:  
[  
\mathcal{L}n^{\text{cont}}\bigl(\psi\_n(\sigma),\partial\sigma \psi\_n(\sigma);{\psi*{n'}(\sigma)}\bigr),  
]  
such that, in the continuum approximation, the discrete Lagrangian satisfies  
[  
\mathcal{L}\_n\bigl(\psi\_n(m+1),\psi\_n(m);\dots\bigr)  
;\approx;  
\Delta\sigma;\mathcal{L}*n^{\text{cont}}\bigl(\psi\_n(\sigma\_m),\partial*\sigma \psi\_n(\sigma\_m);\dots\bigr).   
]

Substituting into (S\_{\text{disc}}) and taking (\Delta\sigma \to 0), the sum over (m) becomes a Riemann sum for an integral over (\sigma):

[  
S\_{\text{disc}}[\Psi]  
;\to;  
S\_{\text{cont}}[\Psi]  
:= \int\_{\sigma\_0}^{\sigma\_1} \mathrm{d}\sigma;  
\sum\_{n\in\mathbb{Z}} g\_n,  
\mathcal{L}*n^{\text{cont}}\bigl(\psi\_n(\sigma),\partial*\sigma \psi\_n(\sigma);{\psi\_{n'}(\sigma)}\bigr).   
]

We may rewrite this as  
[  
S\_{\text{cont}}[\Psi]  
= \int\_{\sigma\_0}^{\sigma\_1} \mathrm{d}\sigma;  
\mathcal{L}*{\text{tot}}(\sigma),  
]  
with  
[  
\mathcal{L}*{\text{tot}}(\sigma)  
:= \sum\_{n} g\_n, \mathcal{L}*n^{\text{cont}}\bigl(\psi\_n(\sigma),\partial*\sigma\psi\_n(\sigma);\dots\bigr).  
]

**(c) Structural properties of the continuum embedding**

A few features of this continuum embedding are important:

1. **Pivot-weighted structure**  
   At each (\sigma), the same pivot weight (g\_n) multiplies all contributions from band (n); this mirrors the discrete construction where (g\_n) scales the entire band contribution (\mathcal{L}\_n).
2. **Localization near the hinge**  
   Since (g\_n) is normalized at the hinge band (n=0) and differs away from it, the context-time integral is often dominated by contributions from bands near the hinge, especially in calibrations where non-hinge (g\_n) are smaller or where (D(n)) is such that the action density is largest near (n=0).
3. **Context-time covariance**  
   The choice of parameter (\sigma) is not essential: any monotone reparametrization (\sigma \mapsto \tilde{\sigma}(\sigma)) with appropriate Jacobian gives the same structure of action, with (\partial\_\sigma) replaced by (\partial\_{\tilde{\sigma}}) and (\mathrm{d}\sigma) by (\mathrm{d}\tilde{\sigma}) times the Jacobian.

The continuum action (S\_{\text{cont}}) provides the starting point for:

* defining **canonical momenta** in context-time,
* deriving **Euler–Lagrange equations** and a **Hamiltonian density**,
* and, after hinge projection and thickening, relating the ladder dynamics to familiar field-equation structures.

**7.2.2 Division-by-Zero Operator & Hinge Projection**

The context-time action (S\_{\text{cont}}) is a **one-dimensional** integral over (\sigma), with an internal sum over ladder bands. To connect this description to a higher-dimensional field-theoretic action (e.g. on a 4-dimensional manifold), we introduce a **formal operator** that:

1. **Localizes** the context-time integral to the hinge region (where the pivot is normalized), and
2. **Thickens** the 2D hinge boundary into a higher-dimensional domain by adding extra coordinates.

This operator is referred to as the **division-by-zero operator**, not because any literal numeric division by 0 is performed, but because it captures the idea of “collapsing a 1D integral to a point and then promoting that point into a higher-dimensional domain.”

**(a) Context-time action as 1D integral**

We start from the continuum action in compact form:  
[  
S\_{\text{cont}}[\Psi]  
= \int\_{\sigma\_0}^{\sigma\_1} \mathrm{d}\sigma;  
\mathcal{L}*{\text{tot}}(\sigma),  
]  
where (\mathcal{L}*{\text{tot}}(\sigma)) is assumed to be sharply peaked around the hinge region (\sigma \approx \sigma\_0) (corresponding, for example, to the contribution of the hinge band (n=0) dominating the sum over (n)).

Formally, as this peak becomes narrower, one may approximate:  
[  
S\_{\text{cont}}[\Psi]  
;\approx; C; \mathcal{L}\_{\text{tot}}(\sigma\_0),  
]  
for some finite constant (C) given by the effective width of the hinge neighborhood in (\sigma).

**(b) Definition of the division-by-zero operator**

We now define a **formal operator** (\mathcal{D}*0^{-1}), the* ***division-by-zero operator****, acting on the context-time integral:  
[  
\mathcal{D}0^{-1} \Bigl[ \int \mathrm{d}\sigma; \mathcal{L}{\text{tot}}(\sigma) \Bigr]  
;:=; \int*{M} \mathrm{d}^4x; \mathcal{L}\_4\bigl(\Phi(x),\partial\Phi(x)\bigr),  
]  
with the following interpretation:

1. **Hinge localization**  
   (\mathcal{D}*0^{-1}) implicitly performs a localization at the hinge:  
   [  
   \int \mathrm{d}\sigma; \mathcal{L}*{\text{tot}}(\sigma)  
   ;\mapsto; \mathcal{L}*{\text{tot}}(\sigma\_0),  
   ]  
   absorbing the finite prefactor (C) into the normalization of the higher-dimensional measure. In symbolic terms, one may think of  
   [  
   \int \mathrm{d}\sigma; \mathcal{L}*{\text{tot}}(\sigma)  
   ;\sim; \Bigl(\int \mathrm{d}\sigma\Bigr),\mathcal{L}*{\text{tot}}(\sigma\_0)  
   ;\to; \frac{\mathcal{L}*{\text{tot}}(\sigma\_0)}{0},  
   ]  
   where the “(1/0)” metaphor indicates that the width of the hinge neighborhood is taken to zero while keeping a finite effective contribution. In practice, (\mathcal{D}\_0^{-1}) is just a compact notation for this limiting procedure.
2. **Hinge evaluation and promotion to fields on (M)**  
   The hinge-evaluated Lagrangian is then promoted to a Lagrangian density on a 4-dimensional manifold (M) by:
   * selecting hinge-level field data (\Phi) (e.g. boundary or pivot fields) from (\Psi(\sigma\_0)),
   * interpreting derivatives in context-time and along the ladder as combinations of derivatives (\partial\_\mu) on (M),
   * defining a 4D Lagrangian density (\mathcal{L}*4(\Phi,\partial\Phi)) that matches (\mathcal{L}*{\text{tot}}(\sigma\_0)) in functional form.
3. **Thickening of the hinge boundary**  
   The hinge band has an effectively 2D boundary (area-law behavior). The division-by-zero step includes a **“thickening”** of this 2D hinge into a 4D domain by introducing additional directions (e.g., a physical time coordinate and a radial or depth coordinate). Symbolically,  
   [  
   \mathcal{D}\_0^{-1} :  
   \quad  
   \text{(2D hinge data + 1D context-time)}  
   ;\longrightarrow;  
   \text{4D field data on } M,   
   ]  
   with the 4D volume measure (\mathrm{d}^4x) absorbing the effective factors associated with the hinge neighborhood and the added directions.

Under this operator, we define the **hinge-projected 4D action**:  
[  
S\_{4D}[\Phi]  
:= \mathcal{D}*0^{-1}\bigl[ S*{\text{cont}}[\Psi] \bigr]  
= \int\_{M} \mathrm{d}^4x; \mathcal{L}\_4(\Phi(x),\partial\Phi(x)).  
]

The mapping (\Psi \mapsto \Phi) and the precise identification of (\partial\_\sigma) and ladder derivatives with combinations of (\partial\_\mu) on (M) are left abstract at this stage; they are fixed when we specify concrete field sectors (e.g. scalar, gauge, gravitational) in later parts of the theory.

**7.2.3 Structural role and limitations**

The continuum embedding and division-by-zero operator play the following **structural roles**:

1. **Context-time to field-theory bridge**  
   (S\_{\text{cont}}) is a one-dimensional (in (\sigma)) action on the context ladder. (\mathcal{D}*0^{-1}) provides a formal bridge to a higher-dimensional action (S*{4D}) on a manifold (M), preserving the functional form of the Lagrangian while localizing at the hinge and “thickening” its boundary.
2. **Hinge fixation**  
   The hinge band (where (D=2), (g=1)) is the natural place to perform this projection: it is the unique area-law band, and its boundary carries the pivot structures (collapse kernels, present plane, inverse-square scaling) that will underlie the 4D fields. The division-by-zero operator is therefore inherently tied to the hinge.
3. **Purely formal object**  
   The operator (\mathcal{D}\_0^{-1}) is treated as a **formal device**:
   * it is not a numeric division operation,
   * it summarizes a two-step procedure (localize at the hinge, then promote to a higher-dimensional domain),
   * its only requirements are internal consistency with the hinge normalization and the dimensional structure of the theory.

In summary, Section 7.2 has:

* Embedded the discrete ladder action into a **continuum context-time action** (S\_{\text{cont}}), with pivot-weighted band contributions.
* Introduced a **division-by-zero operator** (\mathcal{D}*0^{-1}) that encodes hinge localization and dimensional promotion, yielding a higher-dimensional action (S*{4D}) from (S\_{\text{cont}}).

In the next section (7.3), we will use the continuum action (S\_{\text{cont}}) to define **canonical momenta** and a **Hamiltonian density in context-time**, and derive the corresponding **equations of motion** for the ladder state (\Psi(\sigma)).

## **7.3 Hamiltonian Density & Equations of Motion**

Section 7.2 gave a continuum action in **context-time** (\sigma),  
[  
S\_{\text{cont}}[\Psi]  
= \int\_{\sigma\_0}^{\sigma\_1} \mathrm{d}\sigma;  
\sum\_{n} g\_n, \mathcal{L}*n^{\text{cont}}\bigl(\psi\_n(\sigma),\partial*\sigma\psi\_n(\sigma);{\psi\_{n'}(\sigma)}\bigr),  
]  
where (\Psi(\sigma) = {\psi\_n(\sigma)}\_{n\in\mathbb{Z}}) is the ladder state and (g\_n = g(D(n))) are the bandwise pivot weights.

We now:

* define **canonical momenta** with respect to context-time,
* construct the **Hamiltonian density** for the ladder,
* and write down the corresponding **equations of motion** (EOM) in Hamiltonian and Lagrangian form.

Throughout, remember that (\sigma) is *context-time*, not physical time; the resulting Hamiltonian generates evolution along the context ladder’s internal “time” direction.

**7.3.1 Canonical momenta in context-time**

We treat each (\psi\_n(\sigma)) as a (possibly multi-component) field on the boundary graph (\mathcal{G}*n). To make formulas concrete, think of (\psi\_n(\sigma)) as a finite- or infinite-dimensional vector whose components we denote by (\psi*{n,\alpha}(\sigma)), where (\alpha) indexes nodes or modes of (\mathcal{H}\_n^\partial).

The total Lagrangian at context-time (\sigma) is  
[  
\mathcal{L}*{\text{tot}}(\sigma)  
:= \sum*{n} g\_n, \mathcal{L}*n^{\text{cont}}\bigl(\psi\_n(\sigma),\partial*\sigma\psi\_n(\sigma);{\psi\_{n'}(\sigma)}\bigr).  
]

**Definition 7.3.1 (Canonical momenta in context-time).**  
For each band (n) and component index (\alpha), the **canonical momentum** conjugate to (\psi\_{n,\alpha}(\sigma)) is  
[  
\pi\_{n,\alpha}(\sigma)  
:= \frac{\partial \mathcal{L}*{\text{tot}}(\sigma)}  
{\partial\bigl(\partial*\sigma \psi\_{n,\alpha}(\sigma)\bigr)}.  
]

Collectively we write  
[  
\pi\_n(\sigma) := \bigl(\pi\_{n,\alpha}(\sigma)\bigr)\_\alpha,  
]  
so each band (n) carries a pair (\bigl(\psi\_n(\sigma),\pi\_n(\sigma)\bigr)), analogous to a field and its conjugate momentum in ordinary field theory.

Because (\mathcal{L}*{\text{tot}}) is a sum of band contributions weighted by (g\_n), we can also write  
[  
\pi*{n,\alpha}(\sigma)  
= g\_n,\frac{\partial \mathcal{L}*n^{\text{cont}}}  
{\partial\bigl(\partial*\sigma \psi\_{n,\alpha}(\sigma)\bigr)},  
]  
plus any cross-band contributions if (\mathcal{L}*n^{\text{cont}}) couples (\partial*\sigma \psi\_n) to neighboring bands. The precise form depends on the chosen Lagrangian, but the definition above is general.

**7.3.2 Hamiltonian density for the ladder**

With canonical momenta defined, we construct the **Hamiltonian density in context-time**, (\mathcal{H}\_{\text{tot}}(\sigma)), by a Legendre transform.

**Definition 7.3.2 (Total Hamiltonian density).**  
The **Hamiltonian density in context-time** is  
[  
\mathcal{H}*{\text{tot}}(\sigma)  
:= \sum*{n,\alpha} \pi\_{n,\alpha}(\sigma),\partial\_\sigma \psi\_{n,\alpha}(\sigma)  
;-; \mathcal{L}\_{\text{tot}}(\sigma).  
]

Equivalently, in band notation,  
[  
\mathcal{H}*{\text{tot}}(\sigma)  
= \sum*{n} \Bigl\langle \pi\_n(\sigma),\partial\_\sigma\psi\_n(\sigma)\Bigr\rangle  
- \sum\_{n} g\_n,\mathcal{L}*n^{\text{cont}}\bigl(\psi\_n,\partial*\sigma\psi\_n;{\psi\_{n'}}\bigr),  
]  
where (\langle \cdot,\cdot\rangle) denotes the natural pairing between the boundary state space (\mathcal{H}\_n^\partial) and its dual.

The **Hamiltonian in context-time**, (H\_{\text{tot}}), is then  
[  
H\_{\text{tot}}[\Psi(\sigma),\Pi(\sigma)]  
:= \mathcal{H}\_{\text{tot}}(\sigma),  
]  
viewed as a functional of the bandwise fields ({\psi\_n(\sigma)}) and momenta ({\pi\_n(\sigma)}) at fixed (\sigma).

In this formulation:

* (H\_{\text{tot}}) generates evolution in (\sigma),
* The ladder state evolves according to Hamilton-like equations with (\sigma) playing the role of time.

**7.3.3 Hamiltonian equations of motion in context-time**

The **Hamiltonian equations of motion** follow from the usual variational principle applied to (S\_{\text{cont}}), but can be written directly in canonical form:

**Proposition 7.3.3 (Hamiltonian EOM in context-time).**  
For each band (n) and component index (\alpha),  
[  
\partial\_\sigma \psi\_{n,\alpha}(\sigma)  
= \frac{\partial \mathcal{H}*{\text{tot}}}{\partial \pi*{n,\alpha}(\sigma)},  
]  
[  
\partial\_\sigma \pi\_{n,\alpha}(\sigma)  
= -\frac{\partial \mathcal{H}*{\text{tot}}}{\partial \psi*{n,\alpha}(\sigma)}.  
]

Collectively,  
[  
\partial\_\sigma \psi\_n(\sigma) = \frac{\delta H\_{\text{tot}}}{\delta \pi\_n(\sigma)},\quad  
\partial\_\sigma \pi\_n(\sigma) = -\frac{\delta H\_{\text{tot}}}{\delta \psi\_n(\sigma)},  
]  
where (\delta/\delta(\cdot)) denotes functional derivatives with respect to the bandwise fields.

These equations describe **how the ladder state moves along context-time** under the influence of the master action. They are entirely structural: no specific choice of (\mathcal{L}\_n^{\text{cont}}) is required to write them down.

**7.3.4 Lagrangian form: Euler–Lagrange equations on the ladder**

The same dynamics can be expressed in terms of **Euler–Lagrange equations** for the fields (\psi\_n(\sigma)), obtained by varying (S\_{\text{cont}}) directly with respect to (\psi\_n).

**Proposition 7.3.4 (Euler–Lagrange equations for band fields).**  
For each band (n) and component index (\alpha),  
[  
\frac{\partial}{\partial \sigma}  
\left(  
\frac{\partial \mathcal{L}*{\text{tot}}}  
{\partial\bigl(\partial*\sigma \psi\_{n,\alpha}(\sigma)\bigr)}  
\right)  
;-;  
\frac{\partial \mathcal{L}*{\text{tot}}}  
{\partial \psi*{n,\alpha}(\sigma)}  
= 0.  
]

Using the definition of (\pi\_{n,\alpha}(\sigma)), this can be written  
[  
\partial\_\sigma \pi\_{n,\alpha}(\sigma)  
= \frac{\partial \mathcal{L}*{\text{tot}}}  
{\partial \psi*{n,\alpha}(\sigma)}.  
]

This is equivalent to the Hamiltonian form, provided the Legendre transform is non-degenerate in (\partial\_\sigma \psi\_n).

Note that:

* Cross-band couplings enter via the dependence of (\mathcal{L}\_n^{\text{cont}}) on neighboring bands’ fields, which leads to coupled equations across different (n).
* The pivot weights (g\_n) appear multiplicatively in (\mathcal{L}\_{\text{tot}}) and thus influence the relative “strength” of each band’s contribution to the EOM.

**7.3.5 Constraints and neutral directions**

Because the AR formalism incorporates **neutral moves** and **gauge-like redundancies** (both at the tick level and the ladder level), the Hamiltonian structure typically includes **constraints**:

* some combinations of (\psi\_n,\pi\_n) correspond to physically equivalent states,
* some canonical momenta can vanish or be determined by constraints rather than arbitrary initial data.

Structurally:

* Neutral words in the flip algebra (Section 2.4) and neutral ladder composites (Section 6.2) correspond to **symmetry directions** in the space of ladder histories.
* Variations along these directions do not change (S\_{\text{cont}}), leading to **primary constraints** among (\psi\_n,\pi\_n).
* A full Dirac-style constraint analysis is possible but not carried out in this core volume; it is enough to note that:
  + the Hamiltonian formulation naturally accommodates such constraints,
  + physical observables must be invariant under neutral transformations.

Thus, the Hamiltonian system is not necessarily unconstrained; instead, it reflects the **present-act plus ladder symmetries** embedded in the AR structure.

**7.3.6 Hinge-band equations and pivot simplification**

At the **hinge band** (n=0), several simplifications occur:

* The pivot weight (g\_0 = 1); the hinge Lagrangian (\mathcal{L}\_0^{\text{cont}}) enters the equations without additional scaling.
* The boundary graph (\mathcal{G}\_0) has effective dimension (D(0)=2) and area-law scaling, which often leads to **simpler forms** of (\mathcal{L}\_0^{\text{cont}}) (e.g., 2D-like kinetic or surface terms).
* Collapse and expansion to/from neighboring bands ((n=\pm1)) can often be treated perturbatively, with corrections parameterized by deviations (D(\pm1)-2).

The hinge-band equations are then:

* Hamiltonian form:  
  [  
  \partial\_\sigma \psi\_0(\sigma) = \frac{\delta H\_{\text{tot}}}{\delta \pi\_0(\sigma)},\quad  
  \partial\_\sigma \pi\_0(\sigma) = -\frac{\delta H\_{\text{tot}}}{\delta \psi\_0(\sigma)},  
  ]
* Lagrangian form:  
  [  
  \frac{\partial}{\partial \sigma}  
  \left(  
  \frac{\partial \mathcal{L}*{\text{tot}}}{\partial(\partial*\sigma \psi\_0(\sigma))}  
  \right)
  + \frac{\partial \mathcal{L}\_{\text{tot}}}{\partial \psi\_0(\sigma)}  
    = 0.  
    ]

Because the hinge band is the primary pivot for later field-theoretic constructions (via the division-by-zero operator and hinge thickening), these equations are the ones most directly mapped into **4D field equations** in subsequent parts of the theory.

**7.3.7 Summary**

Section 7.3 has:

* Defined **canonical momenta** (\pi\_n(\sigma)) conjugate to bandwise boundary states (\psi\_n(\sigma)) in context-time.
* Constructed the **Hamiltonian density** (\mathcal{H}*{\text{tot}}(\sigma)) by Legendre transform of the total Lagrangian (\mathcal{L}*{\text{tot}}(\sigma)).
* Written down **Hamiltonian equations of motion** for the ladder state in context-time, and the equivalent **Euler–Lagrange equations** for band fields (\psi\_n(\sigma)).
* Noted the role of **constraints and neutral directions**, reflecting symmetries and redundancies in the AR formalism.
* Highlighted the **hinge band** (n=0) as a simplified pivot sector whose equations are most directly related to emergent 4D field equations.

In the next subsection (7.4), we will promote this Hamiltonian structure to a **canonical quantization** in context-time, define the corresponding **path-sum** (path-integral-like) representation over ladder histories, and show how the AR formalism yields quantum-like superpositions and amplitudes at the context level.

**7.4 Canonical Quantization & Path-Sum Representation**

With the context-time Hamiltonian structure in place, we can now describe **quantization on the ladder**. In the V1 framework, this is done at the level of the **context-time variable** (\sigma) and the **band fields** (\psi\_n(\sigma)), (\pi\_n(\sigma)):

* We promote (\psi\_n,\pi\_n) to operators on a context Hilbert space.
* We write a **Schrödinger-like evolution equation in (\sigma)**.
* We introduce a **path-sum (path-integral-like) representation** over ladder histories, consistent with flip-count and neutral-move equivalence.

This section stays at the structural level: we do not choose a specific representation (e.g., coordinate vs momentum) or specific Lagrangian; we only fix the general form of the quantization.

**7.4.1 Ladder Hilbert space & operator-valued fields**

We first introduce a Hilbert space on which the band fields act as operators.

**Definition 7.4.1 (Ladder Hilbert space).**  
Let (\mathcal{H}\_\text{ladder}) be a Hilbert space that carries representations of the bandwise boundary fields. We promote:

* (\psi\_n(\sigma) \mapsto \hat{\psi}\_n(\sigma)),
* (\pi\_n(\sigma) \mapsto \hat{\pi}\_n(\sigma)),

as operator-valued fields on (\mathcal{H}\_\text{ladder}).

To keep the notation simple, we suppress component indices (\alpha); all statements apply component-wise.

We impose **equal-(\sigma)** canonical commutation (or anti-commutation) relations, depending on whether the band fields are bosonic or fermionic in a given sector. For bosonic-type fields, the canonical commutation relations take the form:

[  
[\hat{\psi}*n(\sigma),\hat{\psi}*{n'}(\sigma)] = 0,  
]  
[  
[\hat{\pi}*n(\sigma),\hat{\pi}*{n'}(\sigma)] = 0,  
]  
[  
[\hat{\psi}*n(\sigma),\hat{\pi}*{n'}(\sigma)] = i,\delta\_{nn'}, \mathbf{1},  
]

where (\delta\_{nn'}) is the Kronecker delta on bands, and the right-hand side is understood as the identity on (\mathcal{H}\_\text{ladder}).

For fermionic-type fields, these commutators are replaced by anti-commutators in the usual way.

The precise split between bosonic and fermionic sectors is left unspecified; the structure supports both.

**7.4.2 Context-time Schrödinger equation**

In the Heisenberg picture, operators carry the (\sigma)-dependence and states (|\Psi\rangle) are fixed; in the Schrödinger picture, states depend on (\sigma) and operators are evaluated at a reference (\sigma). We adopt a Schrödinger-like notation to emphasize the context-time evolution.

**Definition 7.4.2 (Context-time evolution operator).**  
Let (H\_{\text{tot}}) be the Hamiltonian functional derived from (\mathcal{H}*{\text{tot}}(\sigma)). We define the* ***context-time evolution operator*** *(U(\sigma\_2,\sigma\_1)) acting on (\mathcal{H}*\text{ladder}) by

[  
i,\partial\_{\sigma\_2} U(\sigma\_2,\sigma\_1)  
= H\_{\text{tot}}(\sigma\_2), U(\sigma\_2,\sigma\_1),\quad  
U(\sigma\_1,\sigma\_1) = \mathbf{1}.  
]

A (Schrödinger-picture) **ladder state** (|\Psi(\sigma)\rangle) then evolves via

[  
|\Psi(\sigma\_2)\rangle = U(\sigma\_2,\sigma\_1),|\Psi(\sigma\_1)\rangle,  
]

with the **context-time Schrödinger equation**

[  
i,\partial\_\sigma |\Psi(\sigma)\rangle  
= H\_{\text{tot}}(\sigma),|\Psi(\sigma)\rangle.  
]

Here (H\_{\text{tot}}(\sigma)) is obtained from the continuum Hamiltonian density (\mathcal{H}\_{\text{tot}}(\sigma)) by integrating/summing over band degrees of freedom (and any internal indices) at fixed (\sigma). The explicit form depends on (\mathcal{L}\_n^{\text{cont}}), but the structural equation above holds for any such choice.

**7.4.3 Path-sum over ladder histories**

We now express transition amplitudes between ladder states as **path sums** over histories of the band fields in context-time, analogous to a path integral in ordinary quantum field theory.

Let (|\Psi\_{\mathrm{in}}\rangle) and (|\Psi\_{\mathrm{out}}\rangle) be initial and final ladder states at context-times (\sigma\_{\mathrm{in}}) and (\sigma\_{\mathrm{out}}). The transition amplitude is

[  
\mathcal{A}\bigl(\Psi\_{\mathrm{out}},\sigma\_{\mathrm{out}};\big|;\Psi\_{\mathrm{in}},\sigma\_{\mathrm{in}}\bigr)  
:= \langle \Psi\_{\mathrm{out}}|, U(\sigma\_{\mathrm{out}},\sigma\_{\mathrm{in}}),|\Psi\_{\mathrm{in}}\rangle.  
]

In a path-sum representation, this is written as

[  
\mathcal{A}  
= \int \mathcal{D}\psi, \mathcal{D}\pi;  
\exp!\left(i, S\_{\text{cont}}[\psi,\pi]\right),  
]

where:

* the integral is over all histories ((\psi\_n(\sigma),\pi\_n(\sigma))) with fixed boundary conditions at (\sigma\_{\mathrm{in}}) and (\sigma\_{\mathrm{out}}),
* the action (S\_{\text{cont}}) is the continuum context-time action from Section 7.2, expressed in first-order form if needed,
* (\mathcal{D}\psi,\mathcal{D}\pi) is a formal measure over band fields and momenta.

The important structural features are:

1. **Pivot weighting**  
   The integrand uses the **band-weighted action**: each band’s contribution to (S\_{\text{cont}}) is scaled by its pivot factor (g\_n).
2. **Neutral-move equivalence**  
   Histories that differ only by **neutral moves** (tick-level or ladder-level) have the same action and therefore the same weight in the path sum; they belong to the same physical equivalence class.
3. **Flip-count classes**  
   At a more discrete level, histories can be grouped into **flip-count classes** (Section 2.4): different detailed sequences of flips that share the same flip-count vector (\nu) and ladder structure contribute the same phase (e^{i S(\nu)}). The path sum can therefore be regarded as a sum over such classes, each weighted by the same exponential factor and an associated combinatorial multiplicity.

We do not attempt to rigorously define (\mathcal{D}\psi,\mathcal{D}\pi) in this volume; it is treated as a formal symbol for “sum over ladder histories,” consistent with the algebraic structure already defined.

**7.4.4 Flip-count and band-count representations**

Because AR emphasizes **flip counts** and **band moves** as fundamental, it is natural to rewrite the path-sum in terms of discrete combinatorial data rather than continuous fields.

At a schematic level:

* Each history can be encoded by:
  + a **flip-count profile** (\nu(\sigma)) describing how many times each primitive operator is applied in each context-time slice, and
  + a **band-occupancy pattern** describing which bands are active at which (\sigma).

Under the structural assumptions of the theory:

* The action (S\_{\text{cont}}) can be expressed as a functional (S[\nu,\text{bands}]) that depends only on such combinatorial data (up to neutral equivalence), and
* The path sum becomes  
  [  
  \mathcal{A}  
  = \sum\_{\text{classes }[\nu,\text{bands}]}  
  \mathcal{N}[\nu,\text{bands}],  
  \exp!\bigl(i, S[\nu,\text{bands}]\bigr),  
  ]  
  where (\mathcal{N}) is a combinatorial factor (e.g. counting how many micro-histories fall into a given equivalence class).

This representation reinforces that:

* The **primitive algebra + ladder structure** is the true kinematic substrate,
* Continuous fields (\psi\_n(\sigma)) are a convenient effective representation, and
* The path-sum respects flip-count and neutral-move equivalence, rather than specific micro-sequences.

**7.4.5 Gauge/neutral symmetries in the path-sum**

Neutral moves—operator words and ladder composites that leave physical states unchanged—manifest as **redundancies** in the path-sum:

* Summing over histories that differ only by neutral moves is effectively summing over gauge orbits.
* One can in principle introduce a **gauge fixing** or divide by the volume of the neutral symmetry group, as in standard gauge theories, but this is not developed in full detail here.

Structurally:

* The path-sum is taken over equivalence classes of histories under neutral symmetries.
* Observables are functionals of histories that are **invariant under neutral-transformations**, ensuring that physical quantities do not depend on gauge-like choices in the ladder description.

This mirrors the role of gauge invariance in conventional quantum field theory, but the symmetries here arise from the present-act and ladder algebra rather than from a pre-given spacetime gauge group.

**7.4.6 Summary**

Section 7.4 has:

* Promoted the band fields (\psi\_n(\sigma)) and momenta (\pi\_n(\sigma)) to **operator-valued fields** on a ladder Hilbert space (\mathcal{H}\_\text{ladder}).
* Defined **canonical commutation relations** at equal context-time, and a **context-time Schrödinger equation** governed by the Hamiltonian (H\_{\text{tot}}).
* Introduced a **path-sum representation** of transition amplitudes over ladder histories in context-time, with weights (e^{i S\_{\text{cont}}[\psi,\pi]}).
* Highlighted the role of **flip-count classes** and **neutral symmetries** in organizing the path-sum: physically relevant amplitudes depend on equivalence classes of histories, not on individual micro-sequences.
* Prepared the ground for Section 7.5, where we will connect this context-time quantization to **CS sampling** and the **Born-style rule**, showing how classical-like outcomes emerge at the level of Collective Spheres from the underlying path-sum over ladder histories.

**7.5 CS Sampling & Born-Style Rule Recovery**

Sections 7.1–7.4 built a **quantized ladder** in context-time:

* band fields (\psi\_n(\sigma)), (\pi\_n(\sigma)) promoted to operators,
* a context-time Hamiltonian (H\_{\text{tot}}),
* a Schrödinger equation in (\sigma), and
* a path-sum over ladder histories with weight (\exp(i S\_{\text{cont}})).

In Section 4.6 we defined a **Born-style weighting** at the hinge: present-plane amplitudes on outcome basins ({R\_i}) of the IN attractor satisfy (p\_i = |v\_i|^2), matching a reference measure (\mu\_{\mathrm{IN}}(R\_i)) in equilibrium.

We now show how this Born-style rule is **recovered** from the ladder’s quantum dynamics by:

* projecting ladder histories onto **Collective Spheres (CS)** at the hinge,
* defining a **CS sampling map** from ladder states to hinge amplitude sets ({v\_i}), and
* using **decoherence / coarse-graining** across context-time to justify the diagonal, probabilistic weights (p\_i = |v\_i|^2).

Throughout, we stay purely structural: no specific experiment or dataset is assumed.

**7.5.1 From ladder amplitudes to hinge outcome basins**

Fix a hinge band (n=0) and a corresponding **hinge CS** (Section 1.4), with boundary graph (\mathcal{G}*0) and IN attractor (K*{\mathrm{IN}}^{(0)}).

* Partition (K\_{\mathrm{IN}}^{(0)}) into outcome basins  
  [  
  K\_{\mathrm{IN}}^{(0)} = \bigsqcup\_{i\in I} R\_i,  
  ]  
  as in Section 4.6.
* The **band-0 boundary state** (\psi\_0(\sigma)) at context-time (\sigma) has a representation (for some basis) whose components can be associated with these basins. At the hinge, we attach a present-plane amplitude to each basin,  
  [  
  v\_i(\sigma) \in \mathcal{P},\qquad i\in I,  
  ]  
  and, when identifying (\mathcal{P}\cong\mathbb{C}), we write (v\_i(\sigma) \leftrightarrow a\_i(\sigma)).

Structurally:

* The ladder dynamics in (\sigma) determines how the collection ({v\_i(\sigma)}\_i) evolves.
* The **CS sampling** procedure will read off these amplitudes at specific CS “sampling events” and map them to probabilities for outcomes ({R\_i}).

**7.5.2 CS sampling map**

A **sampling event** is a context in which a hinge CS applies a strong framing operation (Section 2.3, (CT)) over a finite context-time interval, effectively fixing a set of outcomes ({R\_i}) and reading off their weights.

We model this by a **CS sampling map**:

**Definition 7.5.1 (CS sampling map).**  
At a context-time (\sigma^\ast), for a given hinge CS, define the **CS sampling map**  
[  
\mathsf{Sample}*{\mathrm{CS}} : |\Psi(\sigma^\ast)\rangle ;\mapsto; {v\_i}*{i\in I},  
]  
where:

* (|\Psi(\sigma^\ast)\rangle) is the ladder state at context-time (\sigma^\ast) in the Schrödinger picture,
* ({v\_i}\subset\mathcal{P}) are present-plane amplitudes attached to outcome basins ({R\_i}), extracted from the band-0 part of (|\Psi(\sigma^\ast)\rangle) via a fixed scheme (projection onto basis vectors associated with the basins).

We assume:

1. **Normalization constraint**  
   The amplitudes can be normalized so that  
   [  
   \sum\_{i\in I} |v\_i|^2 = 1.  
   ]
2. **Neutral invariance**  
   Histories that differ by neutral moves (tick-level or ladder-level) but lead to the same effective band-0 state yield the same amplitude set ({v\_i}).
3. **Gate compatibility**  
   Any single-context gate factor (g(D(0))) is neutral at the hinge (since (g(2)=1)), so the sampling map is unaffected by gating at band 0.

Thus, CS sampling is a structural map from ladder states to hinge amplitude assignments over outcome basins.

**7.5.3 Decoherence & effective diagonalization in the CS basis**

We now sketch how **decoherence** in context-time makes the CS sampling outcome effectively **diagonal** in the basin basis, so that off-diagonal interference terms do not appear in the long-run frequencies of CS outcomes.

At a structural level:

* The ladder Hamiltonian (H\_{\text{tot}}) couples band 0 to neighboring bands, and within band 0, different basins ({R\_i}) can, in principle, interfere.
* Strong framing (CT) applied repeatedly to the hinge CS, together with coupling to many external degrees of freedom (outer bands, other CSs), tends to **suppress interference terms** between distinct basins when observables are restricted to hinge CS data.

We capture this by the following assumption about **reduced density operators** for the hinge CS.

Let (\hat{\rho}(\sigma)) be the ladder density operator (pure or mixed) at context-time (\sigma), and let (\hat{\rho}\_0^{\mathrm{CS}}(\sigma)) be the **reduced density operator** for the hinge CS (obtained by tracing out all other bands and degrees of freedom).

In the basis ({|R\_i\rangle}\_i) associated with outcome basins (or an associated orthonormal basis of present-plane amplitude vectors), we assume:

**Axiom 7.5.2 (CS decoherence in basin basis).**  
Under repeated strong framing and typical interactions with other bands:

1. Off-diagonal elements of (\hat{\rho}\_0^{\mathrm{CS}}(\sigma)) in the (|R\_i\rangle) basis become negligible for (\sigma) in the sampling interval:  
   [  
   \langle R\_i|\hat{\rho}\_0^{\mathrm{CS}}(\sigma)|R\_j\rangle \approx 0  
   \quad \text{for } i\neq j.  
   ]
2. The diagonal elements are given by squared norms of present-plane amplitudes:  
   [  
   \langle R\_i|\hat{\rho}\_0^{\mathrm{CS}}(\sigma)|R\_i\rangle  
   = |v\_i(\sigma)|^2,  
   ]  
   consistent with the normalization (\sum\_i |v\_i|^2 = 1).

In other words:

* in the CS basin basis, (\hat{\rho}\_0^{\mathrm{CS}}(\sigma)) is effectively diagonal,
* and its diagonal entries match the **Born-style squared norms** of the hinge amplitudes.

**7.5.4 Recovery of the Born-style rule at the CS level**

Given the decohered, diagonal reduced density operator (\hat{\rho}\_0^{\mathrm{CS}}(\sigma^\ast)), the **probability** of obtaining outcome (R\_i) in a CS sampling event at (\sigma^\ast) is

[  
p\_i := \langle R\_i|\hat{\rho}\_0^{\mathrm{CS}}(\sigma^\ast)|R\_i\rangle = |v\_i(\sigma^\ast)|^2.  
]

This matches the **Born-style rule** stated in Section 4.6:

* (p\_i = |v\_i|^2),
* in equilibrium contexts, (p\_i = \mu\_{\mathrm{IN}}(R\_i)), with (\mu\_{\mathrm{IN}}) the hinge IN measure.

Thus, structurally:

1. Quantization on the ladder yields a context-time evolution of (\hat{\rho}(\sigma)) and its reduced hinge CS density operator (\hat{\rho}\_0^{\mathrm{CS}}(\sigma)).
2. Strong framing and environment-like couplings (to other bands and CSs) render the hinge CS effectively diagonal in the basin basis.
3. The diagonal entries are precisely the squared norms of present-plane amplitudes extracted by CS sampling:  
   [  
   p\_i = |v\_i|^2.  
   ]

This recovers the Born-style probability assignment from the ladder’s path-sum dynamics at the CS level.

**7.5.5 Neutrality and context independence**

The Born-style rule is **invariant under neutral moves** and does not depend on arbitrary choices in the ladder description:

* **Tick-level neutral moves** that do not change band-0 hinge states or the amplitudes ({v\_i}) yield the same probabilities (p\_i).
* **Ladder-level neutral composites** of collapse/expansion that act as identity on the band-0 boundary state also leave the sampling amplitudes unchanged.
* **Single-context gating** at the hinge is neutral (since (g(D(0))=1)); only relative differences in (g(D(n\neq 0))) matter, and those affect higher-band dynamics, not the hinge sampling rule itself.

Therefore:

* The CS sampling rule is **purely structural**: it depends only on the hinge amplitudes ({v\_i}) and the basin decomposition ({R\_i}).
* It is **insensitive** to gauge-like redundancies, neutral moves, or representation choices, as long as the mapping (|\Psi(\sigma^\ast)\rangle \mapsto {v\_i}) is defined consistently.

**7.5.6 Summary**

Section 7.5 has:

* Introduced a **CS sampling map** that takes a ladder state (|\Psi(\sigma^\ast)\rangle) to a set of present-plane amplitudes ({v\_i}) on hinge outcome basins ({R\_i}).
* Used **decoherence and strong framing** at the hinge to argue that the reduced hinge CS density operator (\hat{\rho}\_0^{\mathrm{CS}}) becomes effectively diagonal in the basin basis, with diagonal entries (|v\_i|^2).
* Shown that the **Born-style rule** (p\_i = |v\_i|^2) emerges structurally from the quantized ladder dynamics when we restrict attention to hinge CS outcomes.
* Emphasized that this rule is invariant under **neutral moves** and independent of arbitrary representation choices.

With the master action, context-time quantization, and CS-level Born rule in place, the last step in Part VII is to interpret **context flips as renormalization-group (RG) steps** and describe how (D(n)), (g(D(n))), and other effective couplings flow across bands, which we do in Section 7.6.

**7.6 Renormalization Group & UV Behavior**

The context ladder and its master action naturally support a **renormalization-group (RG)** interpretation:

* **Moving along the ladder** (changing band index (n)) corresponds to coarse-graining or refining the description.
* **Integrating out bands** induces a flow of effective parameters such as (D(n)), (g(D(n))), and any other band-local couplings.
* The ladder structure provides an intrinsic **UV cutoff** and a notion of **fixed points** and **phases** of the theory.

In this section we formalize this interpretation at a structural level.

**7.6.1 Context flips as RG steps**

Recall:

* Bands (n) label context levels, with (n<0) inner (finer), (n>0) outer (coarser).
* Collapse operators (K\_{n+1\to n}) and expansion operators (E\_{n\to n+1}) move boundary states between neighboring bands.
* The reproduction kernel (M\_n = K\_{n+1\to n} \circ U\_{n+1} \circ E\_{n\to n+1}) describes how band-(n) boundary structure reproduces via an “up–evolve–down” cycle.

We interpret such operations as **RG steps**:

* A **downward** step (coarse-graining) is:  
  [  
  \psi\_{n+1} \xrightarrow{K\_{n+1\to n}} \psi\_n,  
  ]  
  integrating out band-((n+1)) details while retaining effective information at band (n).
* An **upward** step (refinement) is:  
  [  
  \psi\_n \xrightarrow{E\_{n\to n+1}} \psi\_{n+1},  
  ]  
  resolving band-(n) structure into more detailed band-((n+1)) degrees of freedom.

An RG **flow** is thus a map that takes a set of effective parameters at one band and produces new effective parameters after (conceptually) integrating out nearby bands.

**7.6.2 Flow of (D(n)), (g(D(n))), and other couplings**

We treat the **dimension curve** (D(n)) and the **pivot weights** (g\_n = g(D(n))) as band-dependent couplings. More generally, any band-local part of the Lagrangian can be parameterized by a set of couplings ({\lambda\_a(n)}):

* (D(n)) (geometric IN dimension),
* (D\_{\mathrm{mem}}(n)) (memory dimension),
* (g\_n) (pivot weight),
* (\lambda\_a(n)) (other band-specific parameters, e.g. mass-like terms, interaction strengths).

An RG step typically proceeds by:

1. **Choosing a reference band** (n) as the “observation scale.”
2. **Integrating out** neighboring bands in a shell (e.g. (n\pm 1), (n\pm 2), etc.).
3. **Rescaling** the ladder coordinate and fields to restore a canonical form of the action.

The result is a new set of effective couplings ({D'(n),g'\_n,\lambda'\_a(n)}) at the reference band.

At a structural level, we write the **RG flow equations** schematically as:

[  
D'(n) = D(n) + \beta\_D(n),  
]  
[  
g'\_n = g\_n + \beta\_g(n),  
]  
[  
\lambda'\_a(n) = \lambda\_a(n) + \beta\_a(n),  
]

where the (\beta)-functions (\beta\_D,\beta\_g,\beta\_a) encode how the couplings change under a single coarse-graining step and rescaling. The exact forms of these (\beta)-functions depend on the detailed Lagrangian choices, but the existence of such flows is guaranteed by the ladder’s multi-band structure and the collapse/expansion algebra.

**7.6.3 Fixed points and phases**

A **fixed point** of the ladder RG is a configuration of couplings that is invariant under RG steps (up to trivial rescalings). Structurally, a fixed point satisfies:

[  
\beta\_D(n) = 0,\quad  
\beta\_g(n) = 0,\quad  
\beta\_a(n) = 0,  
]  
for all bands (n) within a relevant range, or at least for a coarse-grained “block” of bands.

Two particularly important classes of fixed points are:

1. **Hinge-centered fixed point**  
   At the hinge band (n=0), we require:  
   [  
   D(0) = 2,\quad g\_0 = 1,\quad D\_{\mathrm{mem}}(0) = 2,  
   ]  
   and we often assume that the hinge’s couplings (\lambda\_a(0)) are stationary under the RG flow. This makes the hinge band a natural **reference fixed point** for the theory.
2. **Asymptotic inner/outer fixed points**  
   For (n\to -\infty) and (n\to +\infty), the dimension profile tends to limiting values:  
   [  
   D(n) \to D\_{\text{in}}>2,\quad D(n)\to D\_{\text{out}}<2,  
   ]  
   and, correspondingly, the couplings (\lambda\_a(n)) may approach band-independent values (\lambda\_a^{\text{in}}) and (\lambda\_a^{\text{out}}). These represent **inner** and **outer phases** of the ladder, with different effective dimensional behavior.

The ladder may therefore exhibit distinct **phases**:

* an inner, volume-like phase governed by (D\_{\text{in}}),
* an outer, filament-like phase governed by (D\_{\text{out}}),
* and a hinge phase where (D=2), acting as a critical or pivot regime.

In more refined analyses, one can define critical exponents and universality classes associated with these phases, but such details go beyond the scope of this core volume.

**7.6.4 UV cutoff and inner-band saturation**

In conventional field theory, the **UV cutoff** is a high-momentum or small-distance scale beyond which the theory is not resolved. In the AR ladder picture:

* **Inner bands** (n \ll 0) correspond to finer and finer contexts.
* The dimension profile (D(n)) approaches a limiting value (D\_{\text{in}}) as (n\to -\infty).
* The boundary state spaces (\mathcal{H}\_n^\partial) and the reproduction kernels (M\_n) also approach a limiting structure.

This implies:

1. There is a **natural UV floor** at some sufficiently negative band (n\_{\text{UV}}), beyond which further inward refinement does not significantly change the effective couplings—i.e., the theory has saturated its inner structure.
2. The ladder’s **discrete nature** and finite memory dimension per band ensure that the number of effectively independent UV modes is finite (or effectively bounded), providing a built-in UV regulator.

As a result:

* The AR framework avoids naive divergences associated with arbitrarily small scales;
* UV behavior is encoded in a finite number of inner bands and their limiting couplings, rather than in a continuum of arbitrarily high momenta.

**7.6.5 IR behavior and outer-band dilution**

Conversely, the **IR** (infrared) behavior corresponds to **outer bands** (n \gg 0):

* As (n) increases, (D(n) \to D\_{\text{out}} < 2), and boundary graphs become more filament-like.
* The memory dimension (D\_{\mathrm{mem}}(n)) decreases, reflecting a reduced capacity to maintain detailed information at large scales.

Structurally:

* Outer bands act as **effective environments** or **reservoirs** that dilute and decohere fine structure from inner bands.
* Fields and couplings seen at large (n) are coarse, averaged versions of hinge and inner structures, with many details washed out.

This corresponds to an inherent IR behavior:

* Fine-grained information does not remain sharply visible at very large context scales;
* Only robust, low-dimensional features (e.g., long-wavelength modes or conserved charges) survive in the outer regime.

Thus, the same ladder that regularizes UV behavior by saturating inner bands also naturally encodes IR behavior via outer-band dilution.

**7.6.6 Emergent continuum vs discrete corrections**

The ladder RG connects the **discrete band structure** with an **emergent continuum** description:

* For a wide range of bands near the hinge, (D(n)) varies slowly, and the bands can be approximated by a continuous radial coordinate (r).
* Effective couplings become smooth functions of (r), and the master action can be written in a continuum form (S\_{\text{cont}} = \int dr,d^4x,\mathcal{L}\_{\text{eff}}(r,x)), which, after hinge projection and thickening, yields continuum 4D field equations.

However, **discrete corrections** remain important:

* In regimes where (D(n)) changes rapidly (e.g., near inner/outer transitions or sharp features), discrete band effects cannot be neglected.
* These corrections can manifest as higher-derivative terms, non-local interactions, or anomalous scaling in the emergent continuum theory.

From the RG viewpoint:

* The **continuum limit** corresponds to a regime where the step size in (n) (or (r)) is small compared to the scale over which couplings vary.
* Discrete corrections encode **residual lattice effects** of the context ladder, which may be relevant at high energies (inner bands) or very large scales (outer bands).

The AR framework therefore provides both:

* an **emergent continuum field theory** at and around the hinge, and
* a systematic way to understand **UV and IR deviations** as effects of the discrete context ladder and its RG structure.

**7.6.7 Summary**

Section 7.6 has:

* Interpreted **context flips** and the collapse/expansion algebra as **RG steps** on the context ladder.
* Introduced **flows** of band-dependent couplings (D(n)), (g\_n), and (\lambda\_a(n)) under coarse-graining and refinement.
* Identified **fixed points** and **phases** of the ladder:
  + inner (volume-like),
  + hinge (area-law, pivot),
  + outer (filament-like).
* Argued that the ladder provides a natural **UV cutoff** (inner-band saturation) and a structured **IR behavior** (outer-band dilution).
* Clarified how an **emergent continuum** description arises near the hinge, with **discrete corrections** encoding the residual structure of the context ladder.

With this, Part VII’s program is complete: we have defined the master action, its continuum embedding, Hamiltonian and quantized structures, the CS-level Born rule, and the RG interpretation of context flips and band flows. These ingredients form the core dynamical backbone of the V1 AR theory, onto which specific field, gauge, and gravitational sectors are built in the subsequent parts.

**8. Context–Unit Dictionary & Hinge Scales**

**8.1 Context ↔ Time Mapping**

Up to this point, the context ladder has been purely **dimensionless**:

* Bands (n\in\mathbb{Z}) (or radial parameter (r)) label inner/outer contexts.
* The invariant interval (\Delta t, \Delta \tau, |\Delta x|) is defined in “internal units” via flip-count functions.
* The master action and RG flows treat (n) and (r) as abstract indices, not as physical units.

In this part we introduce a **symbolic dictionary** that maps context bands to **coarse-grained temporal scales**. We keep this entirely at the formal level:

* no numerical values,
* no empirical calibrations,
* only the structure of how “units of time” can be attached to bands on the ladder.

Later subsections (8.2–8.4) will introduce a spatial hinge length (\ell\_{\mathrm{UGM}}) and a temporal hinge (T^\*), and relate them to the invariant interval. Here we only set up the **time-side context mapping**.

**8.1.1 Temporal scale labels attached to bands**

We introduce an abstract temporal scale attached to each band (n).

**Definition 8.1.1 (Bandwise temporal scale).**  
For each context band (n \in \mathbb{Z}), we associate a positive scalar  
[  
\Theta\_n > 0,  
]  
called the **band-(n) temporal scale**, with:

1. **Hinge normalization**  
   [  
   \Theta\_0 := T^\*,  
   ]  
   where (T^\*) is a distinguished symbolic time scale (the temporal hinge), defined later in more detail (Section 8.3).
2. **Monotonicity**
   * For inner bands (n<0) (finer contexts),  
     [  
     \Theta\_{n} < \Theta\_{n+1},  
     ]  
     i.e. as we move inward (more negative (n)), the associated temporal scales become shorter.
   * For outer bands (n>0) (coarser contexts),  
     [  
     \Theta\_{n} > \Theta\_{n-1},  
     ]  
     so as we move outward (larger (n)), the associated scales become longer.

We do not assign any physical units or numerical values to (\Theta\_n); they are **relative scales** that order bands in time.

**8.1.2 Parametric form and scale factor**

To make the structure more explicit, it is convenient (but not necessary) to introduce a **bandwise scale factor** governing the ratio between successive (\Theta\_n).

**Definition 8.1.2 (Temporal scaling factor).**  
Assume there exists a strictly positive function (\lambda : \mathbb{Z} \to \mathbb{R}*{>0}) such that  
[  
\Theta*{n+1} = \lambda\_n ,\Theta\_n,\quad \lambda\_n := \lambda(n).  
]

Then, by recursion,  
[  
\Theta\_n = \Theta\_0 ,\prod\_{k=0}^{n-1} \lambda\_k \quad (n>0),  
]  
[  
\Theta\_n = \Theta\_0 ,\prod\_{k=n}^{-1} \lambda\_k^{-1} \quad (n<0).  
]

We impose:

1. **Positivity**  
   (\lambda\_n > 0) for all (n).
2. **Monotone ordering consistency**  
   (\lambda\_n > 1) for sufficiently large positive (n) (outer bands expand timescales), and (\lambda\_n < 1) for sufficiently negative (n) (inner bands contract timescales).
3. **Smoothness in the ladder sense**  
   (\lambda\_n) varies slowly with (n), e.g. there exists a constant (L\_\lambda>0) such that  
   [  
   |\lambda\_{n+1} - \lambda\_n| \le L\_\lambda.  
   ]

When needed, we may consider the simplest case where (\lambda\_n) is constant:  
[  
\lambda\_n \equiv \Lambda > 1,  
]  
yielding a geometric progression  
[  
\Theta\_n = T^\* ,\Lambda^n.  
]  
But the core formalism only requires that a monotone, band-dependent scaling factor (\lambda\_n) exist.

**8.1.3 Compatibility with the invariant interval**

The invariant interval (\Delta t, \Delta \tau, |\Delta x|) was defined in terms of flip-counts, without reference to context bands. We now require that the **context-band temporal scales** (\Theta\_n) be consistent with this interval in the following sense:

1. **Bandwise coarse-grained durations**  
   For a given context band (n), we interpret (\Theta\_n) as the **order of magnitude** of a typical proper-time increment (\Delta\tau) associated with “one natural unit” of present-moment dynamics at that band.

More precisely, we assume there exists a family of flip-count vectors (\nu\_n^{(1)}) (one per band) such that  
[  
\Delta \tau(\nu\_n^{(1)}) \sim \Theta\_n,  
]  
up to a band-independent, dimensionless proportionality factor. This does not fix (\Theta\_n) numerically; it only says that band-(n) units of proper time can be compared to the band-(0) unit (T^\*) via the ladder mapping.

1. **Relational definition of “long” vs “short”**  
   If (\Theta\_n \ll \Theta\_0), then events described primarily at band (n) correspond to **shorter proper-time structures** relative to the hinge. If (\Theta\_n \gg \Theta\_0), they correspond to **longer proper-time structures**.

This gives a **relational meaning** to the temporal scale attached to each band, in terms of the invariant interval.

We do **not** specify how (\nu\_n^{(1)}) is chosen (e.g., number of flips, composition of (F,S,T,C,CT)); the mapping is structural and depends on how one decides to define a “unit” at each band.

**8.1.4 Relation to context-time (\sigma)**

Recall that context-time (\sigma) is an abstract parameter used in the master action and Hamiltonian formulation (Sections 7.1–7.3). It is **not** the same as the bandwise temporal scales (\Theta\_n), but we can relate them qualitatively.

We can think of (\sigma) as providing an **ordering** of ladder configurations, while the bandwise (\Theta\_n) provide a **scale** for how much proper time each band’s dynamics represents.

Structurally:

* For a given band (n), one context-time increment (\Delta\sigma) corresponds to an **effective physical duration** of order (\Theta\_n) when the dynamics is dominated by that band’s contributions.
* A reparametrization of (\sigma) that rescales the effective density of updates at band (n) can be absorbed into a rescaling of (\Theta\_n) or vice versa, as long as the **product** (number of context-time steps × bandwise time scale) reflects the same total proper-time interval in the invariant sense.

In other words:

(\sigma) tracks “how many update steps” have occurred along the ladder, while (\Theta\_n) tells us “what time-units those updates correspond to” at band (n).

No unique mapping is fixed here; the dictionary simply provides a way to phrase bandwise dynamics in terms of abstract temporal units.

**8.1.5 Hinge as matching point for internal and external time**

The hinge band (n=0) is where several independent structures intersect:

* The inner dimension (D(0) = 2) (area-law boundary).
* The pivot weight (g(D(0)) = 1).
* The present exponent (d\_{\text{PMS}}(0) = 2).
* The memory dimension (D\_{\mathrm{mem}}(0) = 2).
* The hinge temporal scale (\Theta\_0 = T^\*).

The **context ↔ time mapping** uses this hinge to anchor the dictionary:

1. Internal: At the hinge, **one natural unit** of band-(0) dynamics corresponds to a proper-time increment (\Delta\tau) of order (T^\*), in the sense of (\Delta\tau(\nu\_0^{(1)}) \sim T^\*) for some canonical flip-count vector (\nu\_0^{(1)}).
2. External: When this theory is eventually compared to a physical model of time (e.g. SI seconds or some other unit), one would map (T^\*) to a physical duration; however, such calibration lies outside this purely theoretical volume and is deferred to the empirical/interpretive volumes.

Thus, the hinge provides the **matching point** between:

* the internal notion of time derived from flip counts and the ladder, and
* any external, unit-bearing notion of time one may want to introduce.

Here we only define the **internal side**: (\Theta\_n) and (T^\*) as abstract temporal scales linked to band index (n) and the invariant interval.

**8.1.6 Summary**

Section 8.1 has:

* Attached a **symbolic temporal scale** (\Theta\_n > 0) to each context band (n), with hinge normalization (\Theta\_0 = T^\*).
* Introduced a **bandwise scaling factor** (\lambda\_n) such that (\Theta\_{n+1} = \lambda\_n \Theta\_n), with monotone behavior (inner bands finer, outer bands coarser).
* Required **compatibility** between (\Theta\_n) and the invariant interval, via band-dependent “unit” flip-counts (\nu\_n^{(1)}) whose proper-time increments are of order (\Theta\_n).
* Clarified the qualitative relation between **context-time** (\sigma) and bandwise scales (\Theta\_n): (\sigma) orders updates, (\Theta\_n) scales them.
* Emphasized the hinge band as the **matching point** where the internal time-unit (T^\*) is defined, aligned with the broader hinge structure (D(0)=2), (g(2)=1), (D\_{\mathrm{mem}}(0)=2).

In the next subsection (8.2), we will build the spatial side of this dictionary by introducing a **bandwise spatial scale mapping**, defining a symbolic hinge length (\ell\_{\mathrm{UGM}}), and relating it to the dimension profile (D(n)) in a way that mirrors the temporal mapping presented here.

**8.2 Context ↔ Spatial Scale Mapping & (\ell\_{\mathrm{UGM}})**

In Section 8.1 we attached **temporal scales** (\Theta\_n) to bands (n) and identified a temporal hinge (T^\*). We now do the **spatial side** of the dictionary:

* attach a **symbolic spatial scale** to each band,
* single out a **hinge length** (\ell\_{\mathrm{UGM}}) at (n=0),
* and relate these scales to the dimension profile (D(n)).

Again, everything here is structural and symbolic: no numerical values or empirical fits are assumed.

**8.2.1 Bandwise spatial scales**

For each band (n), we associate a positive scalar representing the characteristic **spatial scale** at which that context “resolves” structure.

**Definition 8.2.1 (Bandwise spatial scale).**  
For each context band (n \in \mathbb{Z}), introduce a positive scalar  
[  
L\_n > 0,  
]  
called the **band-(n) spatial scale**, with:

1. **Hinge normalization**  
   [  
   L\_0 := \ell\_{\mathrm{UGM}},  
   ]  
   where (\ell\_{\mathrm{UGM}}) is a distinguished length scale (the **spatial hinge**).
2. **Monotonicity**
   * For inner bands (n<0) (finer contexts),  
     [  
     L\_n < L\_{n+1},  
     ]  
     so inner bands resolve **smaller** spatial scales.
   * For outer bands (n>0) (coarser contexts),  
     [  
     L\_n > L\_{n-1},  
     ]  
     so outer bands deal with **larger** scales.

The sequence ({L\_n}) therefore orders bands from **small-scale / inner** to **large-scale / outer**, with (\ell\_{\mathrm{UGM}}) as the hinge scale in between.

**8.2.2 Geometric progression and bandwise zoom factor**

As with time, it is convenient to define a **zoom factor** between successive bands.

**Definition 8.2.2 (Spatial zoom factor).**  
Assume there exists a strictly positive function (\kappa : \mathbb{Z} \to \mathbb{R}*{>0}) such that  
[  
L*{n+1} = \kappa\_n, L\_n,\quad \kappa\_n := \kappa(n).  
]

Then:

* For (n>0),  
  [  
  L\_n = \ell\_{\mathrm{UGM}} \prod\_{k=0}^{n-1} \kappa\_k.  
  ]
* For (n<0),  
  [  
  L\_n = \ell\_{\mathrm{UGM}} \prod\_{k=n}^{-1} \kappa\_k^{-1}.  
  ]

We impose:

1. **Positivity**  
   (\kappa\_n > 0) for all (n).
2. **Inner vs outer behavior**
   * For sufficiently negative (n), (\kappa\_n < 1): inner bands zoom in to smaller scales as (n) decreases.
   * For sufficiently positive (n), (\kappa\_n > 1): outer bands zoom out to larger scales as (n) increases.
3. **Ladder smoothness**  
   (\kappa\_n) varies slowly with (n), e.g.  
   [  
   |\kappa\_{n+1} - \kappa\_n| \le L\_\kappa  
   ]  
   for some constant (L\_\kappa > 0).

In the simplest case, one may take (\kappa\_n \equiv \Lambda\_s > 1) constant, giving  
[  
L\_n = \ell\_{\mathrm{UGM}} \Lambda\_s^n,  
]  
but the formalism does not depend on this specific choice.

**8.2.3 Hinge length (\ell\_{\mathrm{UGM}}) as geometric mean scale**

The notation (\ell\_{\mathrm{UGM}}) reflects its origin as a **geometric-mean (GM) hinge** between an inner and an outer spatial span. Structurally, we encode this as:

**Definition 8.2.3 (Geometric-mean hinge property).**  
There exist two reference spatial scales

* (L\_{\text{inner}}): representative of the dominant inner span relevant to the hinge,
* (L\_{\text{outer}}): representative of the dominant outer span relevant to the hinge,

such that  
[  
\ell\_{\mathrm{UGM}}^2 = L\_{\text{inner}}, L\_{\text{outer}}.  
]

This expresses (\ell\_{\mathrm{UGM}}) as a **middle scale** between an inner and an outer regime. In a purely symbolic setting:

* (L\_{\text{inner}}) can be thought of as the “typical size” of objects that are still interpreted as **parts** within the hinge context.
* (L\_{\text{outer}}) can be thought of as the “typical size” of structures that define the **environment** of the hinge context.

The geometric-mean relation indicates that (\ell\_{\mathrm{UGM}}) is the scale at which the transition between “part-like” and “environment-like” occurs in a structurally symmetric way.

**8.2.4 Compatibility with the dimension profile (D(n))**

We now relate the spatial scales (L\_n) to the **dimension profile** (D(n)). Since:

* inner bands (n \ll 0) have (D(n) > 2) (volume-like),
* outer bands (n \gg 0) have (D(n) < 2) (filament-like),
* hinge band (n=0) has (D(0) = 2) (area-law),

we require that the way **volume, area, and length** scale with (L\_n) is qualitatively consistent with (D(n)).

Specifically, suppose we consider a boundary ball of radius (R) measured in **band-(n) units**, i.e. in units of (L\_n). Then:

* The number of boundary sites or measure (A\_n(R)) on (\mathcal{G}\_n) grows as  
  [  
  A\_n(R) \propto R^{D(n)}.  
  ]
* Rewriting (R) in terms of a reference length (\ell\_{\mathrm{UGM}}), we get scaling behavior consistent with  
  [  
  A\_n\Bigl(\frac{L}{L\_n}\Bigr) \propto \Bigl(\frac{L}{L\_n}\Bigr)^{D(n)},  
  ]  
  where (L) is some fixed physical or symbolic length.

Thus:

* At the hinge,  
  [  
  A\_0\Bigl(\frac{L}{\ell\_{\mathrm{UGM}}}\Bigr) \propto \Bigl(\frac{L}{\ell\_{\mathrm{UGM}}}\Bigr)^2,  
  ]  
  representing **area-law scaling** at scale (\ell\_{\mathrm{UGM}}).
* For inner bands, with (D(n)>2), the same (L) corresponds to more “volume-like” scaling.
* For outer bands, with (D(n)<2), the same (L) appears more filamentary.

We do not enforce a rigid function (D(n) = f(L\_n)); instead, we require:

**Axiom 8.2.4 (Scale–dimension coherence).**  
The mapping (n \mapsto (L\_n,D(n))) is such that:

1. As (n) decreases (inner direction), (L\_n) decreases and (D(n)) increases (toward volume-like).
2. As (n) increases (outer direction), (L\_n) increases and (D(n)) decreases (toward filament-like).
3. At (n=0),  
   [  
   L\_0 = \ell\_{\mathrm{UGM}},\quad D(0) = 2,  
   ]  
   making the hinge scale the unique **area-law scale** in the ladder.

This couples the spatial context dictionary to the fractal dimension structure in a consistent way.

**8.2.5 Length units and rescaling freedom**

As with time, ({L\_n}) represent **relative spatial scales**, not absolute lengths:

* A global rescaling  
  [  
  L\_n \mapsto \alpha, L\_n \quad \text{for all } n,  
  ]  
  with (\alpha>0), leaves all dimensionless structure unchanged.
* (\ell\_{\mathrm{UGM}}) itself can be rescaled in this way; only **ratios** (L\_n / \ell\_{\mathrm{UGM}}) are structurally meaningful inside this theory volume.

When connecting to physical units (e.g. meters), one would choose a calibration mapping (\ell\_{\mathrm{UGM}}) to a particular length; that step is explicitly **left to companion empirical/interpretive volumes**.

In this core theory volume, we only care about:

* the **ordering** of ({L\_n}),
* their **ratios**,
* and how they relate to the **dimension profile** (D(n)) and the hinge structure.

**8.2.6 Summary**

Section 8.2 has:

* Introduced a **bandwise spatial scale** (L\_n > 0) for each context band (n), with hinge normalization (L\_0 = \ell\_{\mathrm{UGM}}).
* Defined a **spatial zoom factor** (\kappa\_n) such that (L\_{n+1} = \kappa\_n L\_n), with inner bands zooming in and outer bands zooming out.
* Characterized (\ell\_{\mathrm{UGM}}) as a **geometric-mean hinge** between an inner span (L\_{\text{inner}}) and an outer span (L\_{\text{outer}}):  
  [  
  \ell\_{\mathrm{UGM}}^2 = L\_{\text{inner}}, L\_{\text{outer}}.  
  ]
* Linked the spatial scales (L\_n) to the **dimension profile** (D(n)), requiring that inner bands (smaller (L\_n)) be more volume-like, outer bands (larger (L\_n)) more filament-like, and the hinge band (with (L\_0 = \ell\_{\mathrm{UGM}})) an **area-law pivot** with (D(0)=2).
* Emphasized that ({L\_n}) and (\ell\_{\mathrm{UGM}}) are **relative** scales; absolute units and numerical values are reserved for empirical/interpretive work.

In the next subsection (8.3), we will define the **temporal hinge** (T^\*) more explicitly, connect it to flip-count units in the invariant interval, and clarify how (\ell\_{\mathrm{UGM}}) and (T^\*) jointly anchor the conversion between spatial and temporal units via the constant (c) appearing in the interval relation.

**8.3 Temporal Hinge (T^\*)**

The temporal hinge (T^\*) is the **distinguished proper-time scale** associated with the hinge band (n=0). It is the internal unit of “one complete present-moment act” at the hinge context:

* Sub-(T^\*) durations are treated as **sub-structure** internal to a single act.
* Super-(T^\*) durations are treated as **sequences of acts**.

This section defines (T^\*) in terms of the invariant interval and flip counts, and explains its structural role in the context-time dictionary.

**8.3.1 Conceptual role of (T^\*)**

At a qualitative level, (T^\*) plays three roles:

1. **Hinge time unit**  
   It is the canonical proper-time increment associated with a single “unit” of dynamics at the hinge band (n=0). Band-0 processes are naturally described in multiples of (T^\*).
2. **Boundary between micro-time and macro-time**
   * Intervals shorter than (T^\*) are interpreted as **internal sub-structure** inside one present act, not as separate presents.
   * Intervals longer than (T^\*) are interpreted as **sequences of distinct presents**, i.e. concatenations of hinge-acts.
3. **Matching point to external time**  
   When the theory is eventually compared to an external notion of clock time, (T^\*) is the internal scale that would be mapped to a physical duration (e.g. seconds). In this volume we only define its **internal** meaning; external calibration is deferred.

**8.3.2 Definition via hinge flip-count unit**

We define (T^\*) through the invariant interval functions introduced in Section 3.4.

Recall:

* (\Delta t(\nu)), (\Delta \tau(\nu)), (|\Delta x(\nu)|) are functions of the flip-count vector (\nu).
* They satisfy  
  [  
  \Delta t(\nu)^2 = \Delta \tau(\nu)^2 + c^{-2}|\Delta x(\nu)|^2.  
  ]

We now choose a **canonical hinge flip-unit**.

**Definition 8.3.1 (Hinge flip-unit).**  
There exists a distinguished flip-count vector (\nu\_0^{(1)}) (a “hinge unit word”) with the following properties:

1. It is associated with **band-0 dynamics** in the sense that its dominant contribution to the action and evolution occurs at the hinge band (other bands contribute only as perturbations).
2. The associated proper-time increment  
   [  
   \Delta \tau\_0^{(1)} := \Delta \tau(\nu\_0^{(1)})  
   ]  
   is strictly positive.
3. We **define** the temporal hinge (T^*) as  
   [  
   T^* := \Delta \tau\_0^{(1)}.  
   ]

Thus, (T^\*) is the proper-time associated with “one canonical hinge act” in flip-count space.

The spatial and coordinate-time components for this unit word are then  
[  
\Delta t\_0^{(1)} := \Delta t(\nu\_0^{(1)}),\quad  
|\Delta x\_0^{(1)}| := |\Delta x(\nu\_0^{(1)})|,  
]  
related by the invariant interval.

**8.3.3 Sub-(T^*) vs supra-(T^*) intervals**

Once (T^\*) is fixed, we interpret intervals as follows.

**Definition 8.3.2 (Sub-act and multi-act intervals).**

1. A proper-time increment (\Delta \tau) is **sub-act** if  
   [  
   0 < \Delta \tau < T^\*.  
   ]  
   Such intervals are treated as **internal structure** within a single present-moment act at band 0. They correspond to flip-count vectors (\nu) that describe fine-grained sub-processes, but not a full hinge-unit.
2. A proper-time increment (\Delta \tau) is **multi-act** if  
   [  
   \Delta \tau \gtrsim k,T^\*  
   ]  
   for some integer (k \ge 1), corresponding (at the structural level) to a composition of approximately (k) unit words ((\nu\_0^{(1)})^{\circ k}) (modulo neutral moves and corrections). Such intervals are interpreted as **sequences of present acts**, rather than a single act.

In terms of flip-counts:

* Repetition of the hinge unit (\nu\_0^{(1)}) produces  
  [  
  \nu^{(k)} := \underbrace{\nu\_0^{(1)} + \cdots + \nu\_0^{(1)}}\_{k\ \text{times}},  
  ]  
  so that  
  [  
  \Delta \tau(\nu^{(k)}) = k,\Delta\tau(\nu\_0^{(1)}) = k,T^\*,  
  ]  
  and similarly for (\Delta t) and (|\Delta x|).

Sub-(T^*) structure remains* ***inside*** *what is treated as “one present”; multi-(T^*) structure describes **chains** of presents.

**8.3.4 Relation to bandwise temporal scales (\Theta\_n)**

In Section 8.1 we attached a symbolic temporal scale (\Theta\_n) to each band. We now pin down the hinge value:

**Axiom 8.3.3 (Hinge scale identification).**  
The band-0 temporal scale equals the temporal hinge:  
[  
\Theta\_0 = T^\*.  
]

For other bands, (\Theta\_n) can be expressed relative to (T^*) via the scaling factors (\lambda\_n):  
[  
\Theta\_n = T^* \times  
\begin{cases}  
\displaystyle\prod\_{k=0}^{n-1} \lambda\_k, & n>0,\[6pt]  
\displaystyle\prod\_{k=n}^{-1} \lambda\_k^{-1}, & n<0.  
\end{cases}  
]

Interpretation:

* **Inner bands** ((n<0)): (\Theta\_n < T^\*); they describe sub-act processes at finer temporal resolution.
* **Outer bands** ((n>0)): (\Theta\_n > T^\*); they describe dynamics coarse-grained over many hinge acts.

In all cases, (\Theta\_n) is a **relational scale**: it tells us how the “unit act” at band (n) compares to the canonical act at the hinge.

**8.3.5 Structural role of (T^\*) in the theory**

Summarizing its place in the framework:

1. **Anchor for temporal units**  
   (T^\*) is the **internal standard** of time in the V1 theory. All other bandwise scales (\Theta\_n) and proper-time increments (\Delta\tau(\nu)) are structurally compared to it.
2. **Hinge alignment with other pivots**  
   At band 0, the following quantities all coincide structurally:
   * Inner dimension: (D(0) = 2).
   * Pivot weight: (g(D(0))=1).
   * Memory dimension: (D\_{\mathrm{mem}}(0)=2).
   * Present exponent: (d\_{\mathrm{PMS}}(0)=2).
   * Temporal hinge scale: (\Theta\_0=T^\*).

This makes the hinge the unique context where **space-like**, **time-like**, **memory**, and **amplitude** structures are all centered and normalized.

1. **Bridge to external time**  
   In any future empirical or interpretive volume, one can map (T^\*) to an external time unit (e.g. seconds), thereby transporting the entire ladder of temporal scales (\Theta\_n) into physically meaningful numbers. This mapping is not part of the present formal volume, but the structure here is designed to support it.

**8.3.6 Summary**

Section 8.3 has:

* Defined the **temporal hinge** (T^*) as the proper-time associated with a canonical hinge flip-unit (\nu\_0^{(1)}), via (\Delta\tau(\nu\_0^{(1)}) = T^*).
* Characterized **sub-(T^\*)** intervals as internal sub-structure and **multi-(T^\*)** intervals as chains of hinge acts.
* Identified the band-0 temporal scale (\Theta\_0) with (T^*), and related other bandwise scales (\Theta\_n) to (T^*) through ladder scaling factors.
* Placed (T^\*) alongside the other hinge pivots (dimension, gate, memory, present exponent) as the **time-axis anchor** of the theory.

In the next subsection (8.4), we will relate (\ell\_{\mathrm{UGM}}) and (T^\*) through the invariant interval and the constant (c), clarifying how the AR framework internally connects spatial and temporal units at the hinge.

**8.4 Relation Between (\ell\_{\mathrm{UGM}}), (T^\*), and (c)**

We now connect the **spatial hinge** (\ell\_{\mathrm{UGM}}), the **temporal hinge** (T^\*), and the **conversion constant** (c) that appears in the invariant interval  
[  
\Delta t^2 = \Delta \tau^2 + c^{-2}|\Delta x|^2.  
]  
The goal is purely structural:

* to show how (\ell\_{\mathrm{UGM}}) and (T^\*) together set the *units* in which (c) is expressed,
* to define (c) in terms of **canonical hinge-scale flip-counts**,
* and to clarify the rescaling freedom in these choices.

No numerical calibration or physical identification (e.g. with SI units) is made here.

**8.4.1 Hinge interval recap**

At the hinge band (n=0), we have:

* **Spatial hinge**: (L\_0 = \ell\_{\mathrm{UGM}}).
* **Temporal hinge**: (\Theta\_0 = T^\*).
* **Dimension**: (D(0)=2).
* **Pivot weight**: (g(D(0)) = 1).
* **Invariant interval** (for any flip-count vector (\nu)):  
  [  
  \Delta t(\nu)^2 = \Delta \tau(\nu)^2 + c^{-2}|\Delta x(\nu)|^2.  
  ]

We also chose a **canonical hinge flip-unit** (\nu\_0^{(1)}) with  
[  
\Delta \tau(\nu\_0^{(1)}) = T^\*,\quad  
\Delta t(\nu\_0^{(1)}) = \Delta t\_0^{(1)},\quad  
|\Delta x(\nu\_0^{(1)})| = |\Delta x\_0^{(1)}|.  
]

The values (\Delta t\_0^{(1)}) and (|\Delta x\_0^{(1)}|) are not fixed numerically; they are whatever the algebra of flip-count functions yields for this chosen word.

**8.4.2 Null and timelike hinge units**

To connect (\ell\_{\mathrm{UGM}}), (T^\*), and (c), we consider two canonical hinge-scale flip-count vectors:

1. A **timelike hinge unit** (\nu\_0^{(\tau)}), representing one “rest-like” hinge act:
   * (\Delta \tau(\nu\_0^{(\tau)}) = T^\*),
   * (|\Delta x(\nu\_0^{(\tau)})| \approx 0) (no significant spatial separation at the hinge scale),
   * so  
     [  
     \Delta t(\nu\_0^{(\tau)}) \approx T^\*.  
     ]
2. A **null hinge unit** (\nu\_0^{(0)}), representing a minimal “lightlike” hinge-scale act:
   * (\Delta \tau(\nu\_0^{(0)}) = 0),
   * (|\Delta x(\nu\_0^{(0)})| > 0,\ \Delta t(\nu\_0^{(0)})>0),
   * with the invariant interval condition reducing to  
     [  
     0 = \Delta \tau^2  
     = \Delta t(\nu\_0^{(0)})^2 - c^{-2}|\Delta x(\nu\_0^{(0)})|^2,  
     ]  
     so that  
     [  
     c = \frac{|\Delta x(\nu\_0^{(0)})|}{\Delta t(\nu\_0^{(0)})}.  
     ]

We now **tie** the null hinge unit to the hinge spatial and temporal scales.

**8.4.3 Hinge-normalized definition of (c)**

We impose a structural convention that the null hinge unit corresponds to displacements of **order** the hinge scales:

[  
|\Delta x(\nu\_0^{(0)})| \sim \ell\_{\mathrm{UGM}},\quad  
\Delta t(\nu\_0^{(0)}) \sim T^\*.  
]

“(\sim)” here means: up to band-independent, dimensionless factors that can be absorbed into the unit choices. In the simplest normalization, we choose the canonical null hinge word so that:

[  
|\Delta x(\nu\_0^{(0)})| = \ell\_{\mathrm{UGM}},\quad  
\Delta t(\nu\_0^{(0)}) = T^\*.  
]

Then, from the null condition,  
[  
c = \frac{|\Delta x(\nu\_0^{(0)})|}{\Delta t(\nu\_0^{(0)})}  
= \frac{\ell\_{\mathrm{UGM}}}{T^\*}.  
]

In this **hinge-normalized convention**:

* (\ell\_{\mathrm{UGM}}) and (T^\*) define the **unit ratio** between spatial and temporal increments in which the null hinge act has unit speed (c),
* and the conversion constant (c) is simply  
  [  
  c = \frac{\ell\_{\mathrm{UGM}}}{T^\*}  
  ]  
  in the internal units of the theory.

More generally, if we allow a dimensionless factor (\alpha>0), we could write  
[  
|\Delta x(\nu\_0^{(0)})| = \alpha,\ell\_{\mathrm{UGM}},\quad  
\Delta t(\nu\_0^{(0)}) = \alpha,T^*,  
]  
which still yields the same ratio  
[  
c = \frac{\ell\_{\mathrm{UGM}}}{T^*}.  
]

Thus (c) is structurally **pinned** by the relation between the hinge length and hinge time.

**8.4.4 Rescaling freedom and physical calibration**

Although (c = \ell\_{\mathrm{UGM}}/T^\*) in internal units, there is still a global rescaling freedom:

* A spatial rescaling (L\_n \mapsto \alpha L\_n) induces (\ell\_{\mathrm{UGM}} \mapsto \alpha,\ell\_{\mathrm{UGM}}).
* A temporal rescaling (\Theta\_n \mapsto \beta \Theta\_n) induces (T^\* \mapsto \beta,T^\*).

Under such rescalings, the ratio  
[  
c = \frac{\ell\_{\mathrm{UGM}}}{T^*}  
]  
transforms as  
[  
c \mapsto \frac{\alpha,\ell\_{\mathrm{UGM}}}{\beta,T^*}  
= \frac{\alpha}{\beta}, c.  
]

Therefore:

* Without additional constraints, the **numerical value** of (c) is not fixed by the internal theory alone; it can always be rescaled by changing the relative units of spatial and temporal scales.
* What is **structurally fixed** is the statement that there exists a null hinge unit whose spatial and temporal increments are proportional to (\ell\_{\mathrm{UGM}}) and (T^\*), and for which the invariant interval condition defines (c) as their ratio.

When connecting to physical units (e.g. identifying (c) with a measured speed and (\ell\_{\mathrm{UGM}}, T^\*) with specific meter/second scales), one picks a concrete pair ((\alpha,\beta)) so that the internal (c) matches its empirical numerical value. That calibration lies outside this theoretical volume.

**8.4.5 Relation to timelike hinge units and one-act budgets**

The timelike hinge unit (\nu\_0^{(\tau)}) with (\Delta \tau = T^\*) and (|\Delta x|\approx 0) satisfies, from the invariant interval,  
[  
\Delta t(\nu\_0^{(\tau)})^2  
\approx T^{\*2} + c^{-2}\cdot 0  
= T^{*2},  
]  
so  
[  
\Delta t(\nu\_0^{(\tau)}) \approx T^*.  
]

Thus, in the hinge-normalized units:

* the **rest-like hinge act** (no significant spatial displacement) has (\Delta t \approx T^\*),
* the **null hinge act** (maximal propagation at rate (c)) has (|\Delta x| = \ell\_{\mathrm{UGM}}), (\Delta t = T^\*),
* both acts have the same **coordinate-time span** at the hinge scale, with different allocations between proper-time and spatial separation.

This supports the intuitive picture of a **“one act budget”** at the hinge:

* For each hinge-scale act of duration (\Delta t \approx T^\*), the invariant interval demands that the combination of proper-time and spatial displacement fit into the relation  
  [  
  \Delta t^2 = \Delta \tau^2 + c^{-2}|\Delta x|^2.  
  ]
* At resting-like acts, the budget is mostly in (\Delta \tau); at null acts, the budget is entirely in (|\Delta x|) with (\Delta \tau=0).

The hinge scales (\ell\_{\mathrm{UGM}}) and (T^\*), together with (c), therefore define the **unit budget** for how much “space displacement” and “proper-time progress” can be allocated in one canonical present act, at the pivot context.

**8.4.6 Summary**

Section 8.4 has:

* Used the invariant interval and hinge-scale flip-counts to relate the spatial hinge (\ell\_{\mathrm{UGM}}), the temporal hinge (T^\*), and the conversion constant (c).
* Introduced a **null hinge unit** whose spatial and temporal increments are proportional to (\ell\_{\mathrm{UGM}}) and (T^*), yielding the structural relation  
  [  
  c = \frac{\ell\_{\mathrm{UGM}}}{T^*}  
  ]  
  in internal units.
* Clarified the **rescaling freedom**: (\ell\_{\mathrm{UGM}}) and (T^\*) can be jointly rescaled without altering the structural content; only the ratio matters internally.
* Interpreted hinge-scale timelike and null units as different allocations of a **one-act interval budget** at the pivot, governed by the invariant relation.

In the next subsection (8.5), we will assemble the time and space dictionaries into a concise **coupling dictionary** that relates context bands to symbolic physical scales (temporal and spatial), and summarize how the hinge pivots ((D(0), g(2), D\_{\mathrm{mem}}(0), \ell\_{\mathrm{UGM}}, T^\*, c)) jointly anchor the internal unit system of the theory.

**8.5 Coupling Dictionary (Minimal)**

We now assemble the **time and space dictionaries** into a compact **context–unit coupling dictionary**. The goal is to give a *purely structural* mapping from:

* **context bands** (n),
* to **symbolic temporal scales** (\Theta\_n),
* to **symbolic spatial scales** (L\_n),

and indicate **typical roles** those bands play (inner, hinge, outer). No numerical values or empirical assignments are made.

**8.5.1 Band classes and their roles**

We group bands (n) into three broad classes:

1. **Inner bands** (n \ll 0)
   * **Dimension:** (D(n) > 2) (volume-like).
   * **Temporal scales:** (\Theta\_n \ll T^\*) (sub-act time).
   * **Spatial scales:** (L\_n \ll \ell\_{\mathrm{UGM}}) (sub-hinge length).
   * **Role:** encode **fine-grained internal structure** inside what the hinge band treats as a single present environment—e.g. sub-parts, micro-configurations, internal degrees of freedom.
2. **Hinge band** (n = 0)
   * **Dimension:** (D(0) = 2) (area-law).
   * **Temporal scale:** (\Theta\_0 = T^\*) (one present-act unit).
   * **Spatial scale:** (L\_0 = \ell\_{\mathrm{UGM}}) (hinge length).
   * **Role:** defines the **present environment** itself; all other bands are interpreted relative to this pivot.
3. **Outer bands** (n \gg 0)
   * **Dimension:** (D(n) < 2) (filament-like).
   * **Temporal scales:** (\Theta\_n \gg T^\*) (multi-act, coarse time).
   * **Spatial scales:** (L\_n \gg \ell\_{\mathrm{UGM}}) (super-hinge length).
   * **Role:** encode **ambient structures** that contain or surround the hinge environment—e.g. larger-scale organization, backgrounds, collective fields.

These classes are schematic; in practice there is a ladder of bands with gradually changing properties, but the inner/hinge/outer trichotomy captures the main structural roles.

**8.5.2 Minimal dictionary: band → (time, space, role)**

We summarize the mapping in a minimal dictionary form. For clarity, we write:

* (\Theta\_n) as “time unit at band (n)” (relative to (T^\*)),
* (L\_n) as “length unit at band (n)” (relative to (\ell\_{\mathrm{UGM}})).

**Inner bands (n \ll 0)**

* **Time:** (\Theta\_n \ll T^\*)
  + Band-(n) “one unit” is a **fraction** of a hinge act.
  + Processes at these bands refine sub-act dynamics.
* **Space:** (L\_n \ll \ell\_{\mathrm{UGM}})
  + Band-(n) “one unit” is a **sub-hinge** length.
  + Structures at these bands are **parts inside parts** relative to the hinge environment.
* **Role:**
  + Provide **microstructure** underlying hinge-level present content (e.g. internal configurations, nested subsystems).
  + Carry higher IN dimension (D(n)), more volume-like and dense.

**Hinge band (n = 0)**

* **Time:** (\Theta\_0 = T^\*)
  + One canonical **present-moment act**.
  + Sub-(T^*) intervals are internal structure; multi-(T^*) intervals are sequences of acts.
* **Space:** (L\_0 = \ell\_{\mathrm{UGM}})
  + One canonical **hinge length**.
  + Sub-(\ell\_{\mathrm{UGM}}) scales are “internal parts;” supra-(\ell\_{\mathrm{UGM}}) scales are “larger environment.”
* **Role:**
  + Defines the **centered present context**: the reference band.
  + Pivot for:
    - dimension (D=2),
    - gate normalization (g=1),
    - memory dimension (D\_{\mathrm{mem}}=2),
    - present exponent (d\_{\mathrm{PMS}}=2),
    - hinge scales ((\ell\_{\mathrm{UGM}}, T^\*)),
    - interval conversion constant (c = \ell\_{\mathrm{UGM}}/T^\*).

**Outer bands (n \gg 0)**

* **Time:** (\Theta\_n \gg T^\*)
  + Band-(n) “one unit” spans **many hinge acts**.
  + Processes at these bands describe slow, coarse evolution of larger contexts.
* **Space:** (L\_n \gg \ell\_{\mathrm{UGM}})
  + Band-(n) “one unit” spans **many hinge-lengths**.
  + Structures at these bands are **environments and backgrounds** for the hinge.
* **Role:**
  + Provide **macro/ambient structure** in which hinge-level presents are embedded.
  + Carry lower IN dimension (D(n)), more filamentary and sparse.

**8.5.3 Symbolic physical domains (purely structural)**

Without assigning numbers, we can describe **symbolic domains** associated with broad ranges of (n):

* **Deep inner domain** ((n \ll 0))
  + Temporal scales much shorter than (T^\*).
  + Spatial scales much smaller than (\ell\_{\mathrm{UGM}}).
  + Appropriate for describing **sub-present microstructure**—whatever detailed configuration supports a hinge-level present moment.
* **Present domain** ((n \approx 0))
  + Temporal scales on the order of (T^\*).
  + Spatial scales on the order of (\ell\_{\mathrm{UGM}}).
  + Appropriate for describing the **immediate environment** of the present, where area-law and one-act budgets are most direct.
* **Near-outer domain** ((n) modestly positive)
  + Temporal scales somewhat larger than (T^\*).
  + Spatial scales somewhat larger than (\ell\_{\mathrm{UGM}}).
  + Appropriate for describing **local environments** that change more slowly and span multiple hinge acts.
* **Far-outer domain** ((n \gg 0))
  + Temporal scales much larger than (T^\*).
  + Spatial scales much larger than (\ell\_{\mathrm{UGM}}).
  + Appropriate for describing **very large-scale, slowly varying structure**, which acts as a background for the hinge domain.

These labels are intentionally vague: they give **categories** rather than concrete physical systems. Any later matching to, say, “molecular,” “organism-level,” “planetary,” or “cosmic” scales requires empirical choice of which band(s) correspond to those scales, and is treated in companion volumes.

**8.5.4 Summary**

Section 8.5 has:

* Combined the **band–time** dictionary ((n \mapsto \Theta\_n)) and the **band–space** dictionary ((n \mapsto L\_n)) into a minimal **context–unit coupling dictionary**.
* Organized bands into **inner**, **hinge**, and **outer** classes, each with characteristic temporal and spatial scale behavior and qualitative roles.
* Highlighted the hinge band as the unique pivot where:
  + (D(0)=2),
  + (g(2)=1),
  + (D\_{\mathrm{mem}}(0)=2),
  + (\Theta\_0=T^\*),
  + (L\_0=\ell\_{\mathrm{UGM}}),
  + and (c = \ell\_{\mathrm{UGM}}/T^\*).
* Provided a **purely structural** mapping from context bands to symbolic domains (deep inner / present / near-outer / far-outer), without numerical values or empirical content.

With this, the context–unit dictionary and hinge-scale layer (Part 8) is complete. The core V1 theory now has a fully specified internal unit system anchored at the hinge, ready to be used in the construction of explicit field, gauge, and gravitational sectors in the subsequent parts of the unified monograph.

**9. Gauge Structure & Matter Spectrum**

**9.1 Context Connections & U(1) Link Variables**

In this part we build **gauge structure** and, later, **matter spectra** on top of the context ladder developed in Parts V–VII. We start with the simplest gauge group, U(1), and then generalize to non-Abelian groups in later subsections.

The key idea is:

The boundary graphs (\mathcal{G}\_n) at each band (n) carry **connections** between their nodes, represented by group elements on edges. These are the discrete analogues of gauge connections; their loop products (Wilson loops) encode field strength, and their behavior under node-based phase rotations encodes gauge invariance.

This section introduces:

* U(1) link variables on (\mathcal{G}\_n),
* node-based gauge transformations, and
* Wilson loops as basic gauge-invariant observables.

All constructions are purely formal; no empirical matching is attempted here.

**9.1.1 Boundary graphs revisited**

Recall from Section 6.1:

* At each band (n), we have a boundary graph (\mathcal{G}\_n = (V\_n, E\_n)).
* Nodes (v \in V\_n) represent **boundary patches** at band (n).
* Edges (e = (v,w) \in E\_n) encode adjacency relations between patches.

We now require (\mathcal{G}\_n) to be:

* connected,
* locally finite (each node has finite degree),
* and oriented for the purpose of defining link variables (each edge gets a direction).

Formally, we treat each undirected edge as a pair of oriented edges:  
[  
e = (v,w) \quad \leadsto \quad (v\to w), (w\to v).  
]

Let (E\_n^{\text{or}}) denote the set of oriented edges at band (n).

Boundary states (\psi\_n) in (\mathcal{H}\_n^\partial) may be thought of as functions on nodes (e.g. (\psi\_n: V\_n \to \mathbb{C})) or more elaborate configurations; here we only need that functions on nodes are well-defined.

**9.1.2 U(1) link variables as discrete connections**

We now introduce a U(1) **link variable** on each oriented edge of (\mathcal{G}\_n).

**Definition 9.1.1 (U(1) link variables).**  
At band (n), a U(1) connection is specified by a map  
[  
U\_n : E\_n^{\text{or}} \to \text{U(1)},  
]  
assigning to each oriented edge (e = (v\to w)) a phase  
[  
U\_n(v\to w) = e^{i \theta\_n(v\to w)},  
]  
with the **compatibility condition**  
[  
U\_n(w\to v) = U\_n(v\to w)^{-1} = e^{-i\theta\_n(v\to w)}.  
]

Informally:

* (\theta\_n(v\to w)) is a discrete analogue of the line integral of a gauge potential along the edge from (v) to (w).
* (U\_n(v\to w)) is the associated parallel-transport phase factor.

The collection of all such assignments ({U\_n}) constitutes a U(1) connection on the boundary graph (\mathcal{G}\_n).

**9.1.3 Node fields and gauge transformations**

Let a **complex node field** at band (n) be a function  
[  
\phi\_n : V\_n \to \mathbb{C},  
]  
thought of as a matter field living on boundary nodes.

We define **U(1) gauge transformations** as node-local phase rotations:

**Definition 9.1.2 (U(1) gauge transformation).**  
A gauge transformation at band (n) is given by a map  
[  
\lambda\_n : V\_n \to \text{U(1)},\qquad \lambda\_n(v) = e^{i\alpha\_n(v)},  
]  
which acts on node fields as  
[  
\phi\_n(v) \mapsto \phi\_n'(v) := \lambda\_n(v),\phi\_n(v).  
]

To make edge-based transport consistent with these phase rotations, link variables transform as:

[  
U\_n(v\to w) \mapsto U\_n'(v\to w)  
:= \lambda\_n(v), U\_n(v\to w), \lambda\_n(w)^{-1}.  
]

Equivalently, in angle variables,  
[  
\theta\_n'(v\to w) = \theta\_n(v\to w) + \alpha\_n(v) - \alpha\_n(w) \mod 2\pi.  
]

Thus:

* Node fields transform by **left multiplication** by phases.
* Edge link variables transform by **conjugation** with node phases at their endpoints.

These are the discrete analogues of standard U(1) gauge transformations.

**9.1.4 Parallel transport and discrete covariant derivatives**

Given a connection (U\_n), we can define **parallel transport** of node fields along edges:

* Transport from (v) to (w) is implemented by multiplication by (U\_n(v\to w)):  
  [  
  \phi\_n(w) ;\sim; U\_n(v\to w),\phi\_n(v),  
  ]  
  when the field is “purely transported” along the edge.

More generally, we can define a **discrete covariant difference** on edges:

**Definition 9.1.3 (Covariant difference along an edge).**  
For an oriented edge (e = (v\to w)), define  
[  
(\nabla\_n \phi\_n)(v\to w)  
:= U\_n(v\to w),\phi\_n(v) - \phi\_n(w).  
]

Under a gauge transformation:

* (\phi\_n \mapsto \phi\_n' = \lambda\_n\phi\_n),
* (U\_n \mapsto U\_n') as above,

we find  
[  
(\nabla\_n \phi\_n)'(v\to w)  
= \lambda\_n(w),(\nabla\_n \phi\_n)(v\to w),  
]  
so ((\nabla\_n \phi\_n)(v\to w)) transforms covariantly, picking up the phase at the target node (w).

This is the discrete analogue of a gauge-covariant derivative.

**9.1.5 Wilson loops and field strength**

Gauge-invariant information about the connection is encoded in **Wilson loops**, i.e. products of link variables around closed paths.

Let (C) be a closed path on (\mathcal{G}*n), defined by an ordered sequence of nodes ((v\_0, v\_1, \dots, v*{k-1}, v\_k)) with (v\_k = v\_0); the oriented edges are ((v\_j\to v\_{j+1})), (j=0,\dots,k-1).

**Definition 9.1.4 (Wilson loop).**  
The U(1) Wilson loop associated with (C) at band (n) is  
[  
W\_n(C) := \prod\_{j=0}^{k-1} U\_n(v\_j\to v\_{j+1}) \in \text{U(1)}.  
]

Under gauge transformations:

[  
W\_n'(C)  
= \prod\_{j} \lambda\_n(v\_j) U\_n(v\_j\to v\_{j+1}) \lambda\_n(v\_{j+1})^{-1}  
= \lambda\_n(v\_0) \left( \prod\_j U\_n(v\_j\to v\_{j+1}) \right) \lambda\_n(v\_0)^{-1}  
= W\_n(C),  
]  
so Wilson loops are **gauge-invariant**.

We interpret (W\_n(C)) as the discrete analogue of the phase (\exp(i\oint A\cdot dx)), and its deviation from unity as a measure of **field strength**:

* If (W\_n(C) = 1) for all loops (C), the connection is **pure gauge** (flat).
* Nontrivial (W\_n(C)) encode curvature.

In a local sense, one can define small loops (plaquettes) and regard the phases around them as discrete samples of the field strength at band (n).

**9.1.6 Band structure and ladder connections**

So far we have described U(1) connections **within** a single band (n). To respect the ladder structure, we have two further requirements:

1. **Band independence vs coupling**  
   Each band (n) has its own boundary graph (\mathcal{G}\_n) and connection (U\_n). These may be:
   * treated as independent gauge sectors, or
   * coupled via collapse/expansion operators (K\_{n+1\to n}), (E\_{n\to n+1}), e.g. by requiring compatibility of (U\_{n+1}) and (U\_n) along inter-band edges.

The core V1 formalism does not fix a specific coupling scheme here; it only requires that any cross-band connection structure be consistent with the ladder algebra and gauge transformations.

1. **Hinge band as pivot for U(1)**  
   At the hinge band (n=0), the dimension (D(0)=2) and pivot weight (g\_0=1) make (\mathcal{G}\_0) the natural place to define **surface-based** gauge variables, with area-law scaling. U(1) connections on (\mathcal{G}\_0):
   * are directly tied to the 2D hinge boundary,
   * interact with the hinge collapse kernels developed later, and
   * provide the seed for 4D field structures obtained via the hinge thickening and division-by-zero operator.

In later subsections, we will use the band-dependent pivot weights (g\_n = g(D(n))) to construct **pivot-weighted U(1) actions**, and then extend this structure to non-Abelian groups and matter spectra.

**9.1.7 Summary**

Section 9.1 has:

* Introduced **U(1) link variables** (U\_n(v\to w) \in \text{U(1)}) on oriented edges of each boundary graph (\mathcal{G}\_n).
* Defined **node-local gauge transformations** (\lambda\_n(v)\in\text{U(1)}), with covariant transformation rules for node fields (\phi\_n(v)) and link variables.
* Constructed a **discrete covariant difference** ((\nabla\_n \phi\_n)(v\to w) = U\_n(v\to w)\phi\_n(v)-\phi\_n(w)) that transforms covariantly.
* Defined **Wilson loops** (W\_n(C)) as gauge-invariant products of link variables around closed paths, encoding discrete field strength.
* Clarified how U(1) connections live on each boundary graph and how the hinge band (n=0) provides a natural 2D pivot for gauge structures.

In the next subsection (9.2), we will use these U(1) link variables and bandwise pivot weights (g\_n) to construct a **pivot-weighted U(1) action** on the ladder, which will serve as the Abelian prototype for the gauge and field sectors of the theory.

**9.2 Pivot-Weighted U(1) Action**

We now construct a **bandwise U(1) action** on the context ladder, using the link variables and node fields from Section 9.1, and weight each band’s contribution by its **pivot factor** (g\_n = g(D(n))). This gives the Abelian prototype for the gauge sector in AR.

The structure is:

* At each band (n), define a **local U(1) action** (S\_n^{\mathrm{U(1)}}[\phi\_n,U\_n]) on the boundary graph (\mathcal{G}\_n).
* Assemble these into a **ladder action** using the pivot weights (g\_n):  
  [  
  S^{\mathrm{U(1)}} = \sum\_n g\_n, S\_n^{\mathrm{U(1)}}.  
  ]

No specific physical field is assumed here; (\phi\_n) is a generic complex scalar on nodes, and (U\_n) is a U(1) connection on edges.

**9.2.1 Bandwise matter and gauge fields**

For each band (n):

* The boundary graph is (\mathcal{G}\_n=(V\_n,E\_n)).
* A complex **matter field** lives on nodes:  
  [  
  \phi\_n : V\_n \to \mathbb{C}.  
  ]
* A U(1) **link variable** lives on oriented edges:  
  [  
  U\_n : E\_n^{\mathrm{or}} \to \mathrm{U(1)},\quad  
  U\_n(v\to w) = e^{i\theta\_n(v\to w)},  
  ]  
  with (U\_n(w\to v)=U\_n(v\to w)^{-1}).

Under a band-(n) gauge transformation (\lambda\_n : V\_n\to \mathrm{U(1)}),  
[  
\phi\_n(v) \mapsto \phi\_n'(v) = \lambda\_n(v),\phi\_n(v),  
]  
[  
U\_n(v\to w) \mapsto U\_n'(v\to w) = \lambda\_n(v),U\_n(v\to w),\lambda\_n(w)^{-1}.  
]

The discrete **covariant difference** along an edge is  
[  
(\nabla\_n \phi\_n)(v\to w)  
:= U\_n(v\to w),\phi\_n(v) - \phi\_n(w),  
]  
which transforms covariantly:  
[  
(\nabla\_n \phi\_n)'(v\to w)  
= \lambda\_n(w),(\nabla\_n \phi\_n)(v\to w).  
]

These objects will be used to build a gauge-invariant lattice-type action at each band.

**9.2.2 Matter kinetic term on (\mathcal{G}\_n)**

We define a **gauge-invariant kinetic term** for the node field (\phi\_n) by summing the norm squared of covariant differences over edges.

Let (\mu\_n(v)) be a positive node weight (e.g. a discrete measure proportional to local boundary area), and let (w\_n(v\to w)) be a positive edge weight. Then a band-(n) matter kinetic term is:

[  
S\_{n,\text{kin}}[\phi\_n,U\_n]  
:= \frac{1}{2}\sum\_{(v\to w)\in E\_n^{\text{or}}}  
w\_n(v\to w),\bigl|(\nabla\_n \phi\_n)(v\to w)\bigr|^2.  
]

Under a gauge transformation, ((\nabla\_n \phi\_n)(v\to w)) picks up a phase at (w), so (|(\nabla\_n \phi\_n)(v\to w)|^2) is invariant, and hence (S\_{n,\text{kin}}) is **U(1)-gauge invariant**.

One can optionally include a **mass-like term** for (\phi\_n):  
[  
S\_{n,\text{mass}}[\phi\_n]  
:= \frac{m\_n^2}{2}\sum\_{v\in V\_n} \mu\_n(v),|\phi\_n(v)|^2,  
]  
with a band-dependent parameter (m\_n^2). This term is manifestly gauge invariant, since it involves only (|\phi\_n|^2).

**9.2.3 Gauge-field term from Wilson loops**

Gauge-field dynamics is encoded in Wilson loops. For band (n), let (\mathcal{C}\_n) be a chosen set of **elementary loops** (e.g. minimal cycles or plaquettes) in (\mathcal{G}\_n). The Wilson loop on (C\in\mathcal{C}\_n) is

[  
W\_n(C) := \prod\_{(v\to w)\in C} U\_n(v\to w) \in \mathrm{U(1)}.  
]

We define a band-(n) gauge action by summing deviations of (W\_n(C)) from unity:

[  
S\_{n,\text{gauge}}[U\_n]  
:= \sum\_{C\in\mathcal{C}*n} \beta\_n(C),\bigl(1 - \operatorname{Re} W\_n(C)\bigr),  
]  
where (\beta\_n(C)) are positive weights (e.g. related to an effective coupling). Since each (W\_n(C)) is gauge invariant, (S*{n,\text{gauge}}) is gauge invariant as well.

In the continuum limit of a fine graph (\mathcal{G}\_n), this term approximates an integral of the field strength squared (a Maxwell-like term) on the band-(n) boundary.

**9.2.4 Bandwise U(1) action**

Combining matter kinetic, mass, and gauge terms, we define a **band-(n) U(1) action**:

[  
S\_n^{\mathrm{U(1)}}[\phi\_n,U\_n]  
:= S\_{n,\text{kin}}[\phi\_n,U\_n]  
+ S\_{n,\text{mass}}[\phi\_n]  
+ S\_{n,\text{gauge}}[U\_n].  
]

By construction:

* Each term is **local** on the boundary graph (\mathcal{G}\_n).
* Each term is **U(1)-gauge invariant** under node-local phase rotations.
* Band-specific parameters ({m\_n^2, \beta\_n(C), w\_n, \mu\_n}) are **couplings** that can, in principle, flow under the ladder RG.

We emphasize that this is only one convenient form of (S\_n^{\mathrm{U(1)}}); the core requirement is:

(S\_n^{\mathrm{U(1)}}) is a gauge-invariant functional of (\phi\_n) and (U\_n) constructed from covariant differences and Wilson loops on (\mathcal{G}\_n).

**9.2.5 Ladder U(1) action with pivot weights**

We now assemble the bandwise actions into a single **ladder U(1) action** by summing over bands and inserting pivot weights (g\_n = g(D(n))):

[  
S^{\mathrm{U(1)}}[{\phi\_n,U\_n}*n]  
:= \sum*{n\in\mathbb{Z}} g\_n, S\_n^{\mathrm{U(1)}}[\phi\_n,U\_n].  
]

Key properties:

1. **Gauge invariance**  
   Since each (S\_n^{\mathrm{U(1)}}) is gauge invariant under band-(n) U(1) transformations, and the weights (g\_n) are scalars independent of (\phi\_n,U\_n), the full action (S^{\mathrm{U(1)}}) is gauge invariant under independent U(1) transformations on each band.
2. **Pivot weighting**  
   The weights (g\_n = g(D(n))) modulate how strongly each band contributes:
   * Hinge band (n=0): (g\_0 = g(2) = 1); its U(1) action enters unscaled.
   * Inner bands (n<0) and outer bands (n>0) are weighted according to their IN dimensions (D(n)) and the pivot function (g(D)), encoding the structural role of the hinge as a pivot.
3. **Compatibility with the master action**  
   (S^{\mathrm{U(1)}}) is added as one sector of the overall master action:  
   [  
   S\_{\text{total}} = S\_{\text{ladder core}} + S^{\mathrm{U(1)}} + \dots  
   ]  
   where (S\_{\text{ladder core}}) is the context ladder and present-act action from Part VII, and the ellipsis indicates additional gauge/matter sectors.

**9.2.6 Hinge band as U(1) pivot**

At the **hinge band** (n=0):

* Dimension: (D(0)=2) → area-law boundary.
* Gate weight: (g\_0 = 1).
* Spatial scale: (L\_0 = \ell\_{\mathrm{UGM}}).
* Temporal scale: (\Theta\_0 = T^\*).

The band-(0) U(1) action  
[  
S\_0^{\mathrm{U(1)}}[\phi\_0,U\_0]  
= S\_{0,\text{kin}}[\phi\_0,U\_0]  
+ S\_{0,\text{mass}}[\phi\_0]  
+ S\_{0,\text{gauge}}[U\_0]  
]  
is thus:

* defined on a **2D-like boundary** (the hinge),
* directly tied to the hinge scales ((\ell\_{\mathrm{UGM}},T^\*,c)), via the context–unit dictionary, and
* the natural candidate for the **pivot U(1) sector** that will be thickened into a 4D gauge field by the division-by-zero / hinge-thickening process introduced in Part VII.

Inner and outer bands contribute corrections to this hinge sector via their weighted actions (g\_n S\_n^{\mathrm{U(1)}}), but the hinge is structurally the central U(1) frame.

**9.2.7 Summary**

Section 9.2 has:

* Defined a **bandwise U(1) action** (S\_n^{\mathrm{U(1)}}[\phi\_n,U\_n]) on each boundary graph (\mathcal{G}\_n), built from:
  + covariant differences of node fields,
  + mass-like node terms, and
  + Wilson-loop-based gauge terms.
* Assembled these into a **pivot-weighted ladder U(1) action**  
  [  
  S^{\mathrm{U(1)}} = \sum\_n g\_n, S\_n^{\mathrm{U(1)}}.  
  ]
* Shown that the full U(1) ladder action is **gauge invariant** and structurally consistent with the context ladder and pivot function.
* Highlighted the **hinge band** (n=0) as the unweighted, 2D area-law pivot for the U(1) sector, providing the natural base from which a 4D U(1) field theory can be constructed via hinge thickening.

In the next subsection (9.3), we will extend this construction to **non-Abelian gauge groups** (SU(2), SU(3)), defining non-Abelian link variables, covariant differences, and bandwise Yang–Mills-type actions on the context ladder.

**9.3 Non-Abelian Extensions: SU(2), SU(3)**

We now generalize the U(1) construction to **non-Abelian gauge groups**, focusing on compact matrix groups such as SU(2) and SU(3). The pattern is:

* Replace U(1) phases on edges with **group elements** in SU(N).
* Let node fields transform in some representation (typically the fundamental).
* Define **covariant differences** and **Wilson loops** using matrix products and traces.
* Maintain gauge invariance band-by-band and compatibility with the ladder structure.

This section stays completely general: SU(2) and SU(3) are treated as representative examples of compact non-Abelian groups; no specific physical identifications (e.g. “weak” or “color” sectors) are made here.

**9.3.1 Non-Abelian link variables on (\mathcal{G}\_n)**

Let (G) be a compact Lie group (e.g. SU(2) or SU(3)) with unitary matrix representations. At each band (n) and oriented edge (e = (v\to w)\in E\_n^{\mathrm{or}}), we assign a **link variable** in (G).

**Definition 9.3.1 (Non-Abelian link variables).**  
At band (n), a (G)-connection is given by a map  
[  
U\_n : E\_n^{\mathrm{or}} \to G,  
]  
with  
[  
U\_n(v\to w) \in G,  
]  
and the **compatibility condition**  
[  
U\_n(w\to v) = U\_n(v\to w)^{-1}.  
]

In a matrix representation (e.g. fundamental), each (U\_n(v\to w)) is a unitary matrix with determinant 1 for (G = \mathrm{SU}(N)).

Special cases:

* **SU(2)**: (U\_n(v\to w)) is a (2\times 2) unitary matrix with determinant 1.
* **SU(3)**: (U\_n(v\to w)) is a (3\times 3) unitary matrix with determinant 1.

The collection ({U\_n}) across all edges at band (n) defines a **discrete non-Abelian connection** on (\mathcal{G}\_n).

**9.3.2 Node fields in representations of (G)**

Node fields now take values in a representation space of the group (G).

**Definition 9.3.2 (Matter fields in a representation).**  
Let (R) be a (finite-dimensional) representation of (G) with carrier space (V\_R). A band-(n) **matter field** in representation (R) is a map  
[  
\phi\_n : V\_n \to V\_R,  
]  
so that for each node (v \in V\_n),  
[  
\phi\_n(v) \in V\_R.  
]

Examples:

* For SU(2) in the fundamental, (V\_R \cong \mathbb{C}^2), and (\phi\_n(v)) is a 2-component complex vector (a doublet).
* For SU(3) in the fundamental, (V\_R \cong \mathbb{C}^3), and (\phi\_n(v)) is a 3-component complex vector (a triplet).

We will write the group action on (V\_R) as  
[  
g \cdot \phi \quad \text{or simply}\quad g,\phi,  
]  
with (g\in G), (\phi\in V\_R).

**9.3.3 Gauge transformations: node-based group elements**

Gauge transformations are now maps from nodes to group elements.

**Definition 9.3.3 (Non-Abelian gauge transformation).**  
A gauge transformation at band (n) is a map  
[  
\Lambda\_n : V\_n \to G,\qquad \Lambda\_n(v) \in G,  
]  
which acts on node fields as  
[  
\phi\_n(v) \mapsto \phi\_n'(v) := \Lambda\_n(v),\phi\_n(v),  
]  
where (\Lambda\_n(v)) acts in the chosen representation (R).

On link variables, the transformation rule is  
[  
U\_n(v\to w) \mapsto U\_n'(v\to w)  
:= \Lambda\_n(v),U\_n(v\to w),\Lambda\_n(w)^{-1}.  
]

This is the discrete analogue of the continuum transformation  
[  
A\_\mu(x) \mapsto g(x) A\_\mu(x) g(x)^{-1} + g(x) \partial\_\mu g(x)^{-1}.  
]

Compatibility condition (U\_n(w\to v) = U\_n(v\to w)^{-1}) is preserved automatically since  
[  
U\_n'(w\to v)  
= \Lambda\_n(w),U\_n(w\to v),\Lambda\_n(v)^{-1}  
= \Lambda\_n(w),U\_n(v\to w)^{-1},\Lambda\_n(v)^{-1}  
= \bigl(\Lambda\_n(v),U\_n(v\to w),\Lambda\_n(w)^{-1}\bigr)^{-1}  
= U\_n'(v\to w)^{-1}.  
]

**9.3.4 Covariant differences and parallel transport**

We generalize the U(1) covariant difference to the non-Abelian case by using matrix multiplication in the representation (R).

**Definition 9.3.4 (Non-Abelian covariant difference).**  
For an oriented edge (e=(v\to w)), define  
[  
(\nabla\_n \phi\_n)(v\to w)  
:= U\_n(v\to w),\phi\_n(v) - \phi\_n(w).  
]

Under a gauge transformation:

* (\phi\_n(v) \mapsto \phi\_n'(v) = \Lambda\_n(v)\phi\_n(v)),
* (U\_n(v\to w) \mapsto U\_n'(v\to w) = \Lambda\_n(v)U\_n(v\to w)\Lambda\_n(w)^{-1}),

we find  
[  
\begin{aligned}  
(\nabla\_n \phi\_n)'(v\to w)  
&= U\_n'(v\to w),\phi\_n'(v) - \phi\_n'(w) \  
&= \Lambda\_n(v) U\_n(v\to w)\Lambda\_n(w)^{-1},\Lambda\_n(v)\phi\_n(v) - \Lambda\_n(w)\phi\_n(w) \  
&= \Lambda\_n(w)\bigl( U\_n(v\to w)\phi\_n(v) - \phi\_n(w) \bigr) \  
&= \Lambda\_n(w),(\nabla\_n \phi\_n)(v\to w).  
\end{aligned}  
]

Thus ((\nabla\_n \phi\_n)(v\to w)) transforms covariantly under the group action at the **target node** (w).

Parallel transport from (v) to (w) is encoded as the map (\phi\_n(v) \mapsto U\_n(v\to w)\phi\_n(v)); the covariant difference compares this transported value with (\phi\_n(w)).

**9.3.5 Wilson loops and non-Abelian field strength**

For a closed path (C) on (\mathcal{G}*n) with nodes ((v\_0, v\_1, \dots, v*{k-1}, v\_k)) and (v\_k=v\_0), define the **Wilson loop** as the ordered product  
[  
W\_n(C) := U\_n(v\_{k-1}\to v\_k)\cdots U\_n(v\_1\to v\_2)U\_n(v\_0\to v\_1).  
]

Unlike U(1), here (W\_n(C)) is a matrix in the chosen representation. Under a gauge transformation,  
[  
\begin{aligned}  
W\_n'(C)  
&= \Lambda\_n(v\_0)U\_n(v\_0\to v\_1)\Lambda\_n(v\_1)^{-1}  
\cdot \Lambda\_n(v\_1)U\_n(v\_1\to v\_2)\Lambda\_n(v\_2)^{-1}  
\cdots  
\Lambda\_n(v\_{k-1})U\_n(v\_{k-1}\to v\_k)\Lambda\_n(v\_k)^{-1} \  
&= \Lambda\_n(v\_0)\left( \prod\_{j=0}^{k-1} U\_n(v\_j\to v\_{j+1}) \right)\Lambda\_n(v\_0)^{-1} \  
&= \Lambda\_n(v\_0), W\_n(C),\Lambda\_n(v\_0)^{-1}.  
\end{aligned}  
]

So (W\_n(C)) transforms by **conjugation**, and its **trace** in any representation (R),  
[  
\operatorname{Tr}\_R W\_n(C),  
]  
is **gauge invariant**. This is the standard non-Abelian generalization: Wilson loops up to conjugation encode the field strength; traces of Wilson loops are gauge-invariant observables.

Small loops (e.g. minimal cycles or plaquettes) approximate the non-Abelian field strength in a discrete setting:

* (W\_n(C)\approx \exp(i F\_n \cdot \text{area})) in continuum language,
* with non-commuting field components encoded in the order of the product.

**9.3.6 SU(2) and SU(3) as concrete examples**

Although the formalism works for any compact Lie group (G), we highlight SU(2) and SU(3) as canonical cases:

* **SU(2)**:
  + Group elements: (2\times 2) unitary matrices with determinant 1.
  + Fundamental representation: doublets (\phi\_n(v) \in \mathbb{C}^2).
  + Link variables: (U\_n(v\to w)\in \mathrm{SU}(2)).
  + Wilson loops: (W\_n(C)\in \mathrm{SU}(2)), with observables built from (\operatorname{Tr} W\_n(C)) and higher representation traces.
* **SU(3)**:
  + Group elements: (3\times 3) unitary matrices with determinant 1.
  + Fundamental representation: triplets (\phi\_n(v) \in \mathbb{C}^3).
  + Link variables: (U\_n(v\to w)\in \mathrm{SU}(3)).
  + Wilson loops: (W\_n(C)\in \mathrm{SU}(3)), with observables built from (\operatorname{Tr} W\_n(C)) and traces in higher representations as needed.

In both cases:

* The **bandwise structure** is identical: each band (n) carries its own SU(2) or SU(3) connection and matter fields.
* The **ladder weighting** (using (g\_n = g(D(n)))) and the hinge-thickening procedure later apply uniformly.

We defer any identification of specific **physical charges or multiplets** to a later, more detailed treatment of the matter spectrum; here SU(2) and SU(3) are purely abstract groups acting on node fields.

**9.3.7 Band structure and group choice**

One can, in principle, assign **different gauge groups to different bands**:

* The same group (G) (e.g. SU(2) or SU(3)) for all bands.
* A direct product of groups acting differently on inner vs outer bands.
* Or a larger group broken into subgroups that dominate in different band ranges.

The V1 formalism itself is agnostic about this choice; it only requires that:

* gauge transformations be local on nodes at each band,
* link variables transform appropriately on edges, and
* bandwise actions be gauge invariant.

The **hinge band** (n=0) is again special:

* Its boundary dimension (D(0)=2) and unweighted pivot (g\_0=1) make it a natural candidate for the **dominant non-Abelian gauge sector** that gets thickened into 4D via the division-by-zero operator.
* Inner and outer bands may contribute higher- or lower-dimensional corrections (e.g. running couplings, symmetry-breaking patterns, effective potentials) without altering the core hinge structure.

**9.3.8 Summary**

Section 9.3 has:

* Generalized U(1) link variables to **non-Abelian link variables** (U\_n(v\to w)\in G) (e.g. SU(2), SU(3)) on the boundary graphs (\mathcal{G}\_n).
* Defined node fields (\phi\_n(v)\in V\_R) in representations of (G), with gauge transformations (\Lambda\_n(v) \in G) acting locally on nodes.
* Constructed **non-Abelian covariant differences** ((\nabla\_n \phi\_n)(v\to w) = U\_n(v\to w)\phi\_n(v)-\phi\_n(w)) that transform covariantly.
* Introduced **Wilson loops** (W\_n(C)) as ordered products of link variables around closed paths, transforming by conjugation and yielding gauge-invariant observables via traces.
* Highlighted SU(2) and SU(3) as canonical examples, with node fields in their fundamental representations and Wilson loops in matrix form.

In the next subsection (9.4), we will build **pivot-weighted Yang–Mills-type actions** on the context ladder, using these non-Abelian connections and the pivot weights (g\_n), providing the formal non-Abelian gauge sector of the V1 theory.

**9.4 Pivot-Weighted Yang–Mills Actions**

We now construct **non-Abelian gauge actions** on the ladder, generalizing the U(1) case of Section 9.2 to Yang–Mills–type actions for compact groups such as SU(2) and SU(3). The pattern is:

* At each band (n): define a **lattice Yang–Mills action** (S\_n^{\text{YM}}[U\_n]) on (\mathcal{G}\_n).
* Optionally couple to matter fields (\phi\_n) via covariant differences.
* Assemble a **pivot-weighted** total action  
  [  
  S^{\text{YM}} = \sum\_n g\_n, S\_n^{\text{YM}},  
  ]  
  with (g\_n = g(D(n))).

This section focuses on the gauge-field part; matter-coupling and spectrum are treated in later subsections.

**9.4.1 Bandwise Yang–Mills action**

Fix a compact Lie group (G) (e.g. SU(2), SU(3)) and a unitary matrix representation (\rho) on a vector space (V\_R). At band (n):

* Boundary graph: (\mathcal{G}\_n = (V\_n, E\_n)).
* Oriented edges: (E\_n^{\text{or}}).
* Link variables: (U\_n(v\to w) \in G), with (U\_n(w\to v)=U\_n(v\to w)^{-1}).
* Gauge transformations: (\Lambda\_n(v)\in G), acting as  
  [  
  U\_n(v\to w)\mapsto \Lambda\_n(v),U\_n(v\to w),\Lambda\_n(w)^{-1}.  
  ]

Let (\mathcal{C}\_n) be a chosen set of elementary loops (plaquettes) on (\mathcal{G}\_n). For each loop (C\in\mathcal{C}*n), define the* ***Wilson loop*** *in representation (R):  
[  
W\_n(C) := \rho\bigl(U\_n(v*{k-1}\to v\_k)\cdots U\_n(v\_0\to v\_1)\bigr)\in \mathrm{End}(V\_R),  
]  
with nodes ((v\_0,\dots,v\_k)), (v\_k=v\_0).

Under gauge transformations,  
[  
W\_n(C);\mapsto; \rho(\Lambda\_n(v\_0)),W\_n(C),\rho(\Lambda\_n(v\_0))^{-1},  
]  
so the trace (\operatorname{Tr}\_R W\_n(C)) is gauge invariant.

We define a band-(n) Yang–Mills action by summing plaquette contributions built from these loops.

**Definition 9.4.1 (Bandwise Yang–Mills action).**  
For band (n), the Yang–Mills–type gauge action is  
[  
S\_n^{\text{YM}}[U\_n]  
:= \sum\_{C\in\mathcal{C}\_n} \beta\_n(C),  
\Bigl(1 - \frac{1}{\dim R},\Re \operatorname{Tr}\_R W\_n(C)\Bigr),  
]  
where:

* (\beta\_n(C) > 0) are band- and loop-dependent weights (inverse couplings),
* (\dim R = \dim V\_R),
* (\Re) indicates real part.

Gauge invariance is immediate: (\operatorname{Tr}\_R W\_n(C)) is invariant, so (S\_n^{\text{YM}}) is invariant under all (\Lambda\_n).

In the “small plaquette” / fine-graph limit, this functional approximates the continuum Yang–Mills action (\int \mathrm{tr}(F\_{\mu\nu}F^{\mu\nu})) on the band-(n) boundary.

**9.4.2 Matter–gauge coupling at band (n)**

If we include a matter field (\phi\_n: V\_n\to V\_R) in representation (R), we use the non-Abelian **covariant difference** (Section 9.3):  
[  
(\nabla\_n \phi\_n)(v\to w) = U\_n(v\to w),\phi\_n(v) - \phi\_n(w),  
]  
which transforms covariantly:  
[  
(\nabla\_n \phi\_n)'(v\to w)  
= \Lambda\_n(w),(\nabla\_n \phi\_n)(v\to w).  
]

We can then define a gauge-invariant kinetic term:  
[  
S\_{n,\text{kin}}^{\text{matter}}[\phi\_n,U\_n]  
:= \frac{1}{2}\sum\_{(v\to w)\in E\_n^{\text{or}}}  
w\_n(v\to w),\bigl|(\nabla\_n \phi\_n)(v\to w)\bigr|^2,  
]  
where:

* (w\_n(v\to w) > 0) are edge weights,
* (|\cdot|) is the norm induced by the Hermitian inner product on (V\_R).

This term is gauge invariant because (|(\nabla\_n \phi\_n)(v\to w)|^2) is invariant under the group action at (w).

A band-(n) mass term for (\phi\_n) can be added as  
[  
S\_{n,\text{mass}}^{\text{matter}}[\phi\_n]  
:= \frac{m\_n^2}{2},\sum\_{v\in V\_n} \mu\_n(v),|\phi\_n(v)|^2,  
]  
with node weights (\mu\_n(v)).

Putting this together, a full band-(n) gauge+matter action is  
[  
S\_n^{\text{g+m}}[\phi\_n,U\_n]  
:= S\_n^{\text{YM}}[U\_n]  
+ S\_{n,\text{kin}}^{\text{matter}}[\phi\_n,U\_n]  
+ S\_{n,\text{mass}}^{\text{matter}}[\phi\_n].  
]

Again, the exact choice of (\beta\_n(C), w\_n,\mu\_n,m\_n^2) is left free; they are the band-dependent couplings that can run under the ladder RG.

**9.4.3 Pivot-weighted Yang–Mills action on the ladder**

We now form the **pivot-weighted** Yang–Mills action across all bands:

[  
S^{\text{YM}}[{U\_n}*n]  
:= \sum*{n\in\mathbb{Z}} g\_n, S\_n^{\text{YM}}[U\_n],\quad g\_n = g(D(n)),  
]  
and, with matter,  
[  
S^{\text{g+m}}[{\phi\_n,U\_n}*n]  
:= \sum*{n\in\mathbb{Z}} g\_n, S\_n^{\text{g+m}}[\phi\_n,U\_n].  
]

Properties:

1. **Gauge invariance**  
   Each (S\_n^{\text{g+m}}) is invariant under independent gauge transformations (\Lambda\_n: V\_n\to G). Since (g\_n) are scalar weights independent of fields, the total action (S^{\text{g+m}}) is invariant under the direct product of bandwise gauge groups.
2. **Pivot weighting**  
   The weights (g\_n = g(D(n))) modulate the relative contributions of different bands, with the hinge band (n=0) normalized:  
   [  
   g\_0 = g(D(0)) = g(2) = 1.  
   ]  
   Inner and outer bands are scaled according to their IN dimensions and the pivot profile.
3. **Consistency with ladder RG and hinge thickening**  
   This action can be inserted into the ladder’s master action; its band couplings (\beta\_n,m\_n,\dots) can flow under ladder RG, and the hinge band’s contribution will be singled out when performing hinge projection and 4D thickening.

**9.4.4 Hinge band and 4D Yang–Mills sector (structural)**

At the **hinge band** (n=0):

* The boundary graph (\mathcal{G}\_0) is effectively 2D ((D(0)=2)), with area-law scaling.
* The pivot weight (g\_0=1) makes (S\_0^{\text{YM}}) and any matter terms appear **unweighted** in (S^{\text{g+m}}).
* The context–unit dictionary provides hinge scales ((\ell\_{\mathrm{UGM}}, T^\*, c)).

Under the division-by-zero / hinge-thickening procedure (Section 7.2):

* The band-0 Yang–Mills action is promoted to a **4D Yang–Mills action** on a manifold (M), with fields derived from hinge-level link variables and matter fields.
* Inner/outer band contributions (g\_n S\_n^{\text{YM}}) show up as corrections to the hinge 4D sector, e.g. in effective couplings, higher-order operators, or anomalous scaling, depending on how the ladder RG is implemented.

Thus, the pivot-weighted Yang–Mills action provides the structural backbone for **emergent 4D non-Abelian gauge fields** in the AR framework, with the hinge band as the primary source.

**9.4.5 Summary**

Section 9.4 has:

* Defined a **bandwise non-Abelian Yang–Mills action** (S\_n^{\text{YM}}[U\_n]) on each boundary graph (\mathcal{G}\_n), using Wilson loops and traces in a representation of (G).
* Included optional **matter–gauge coupling** via covariant differences and mass terms, forming (S\_n^{\text{g+m}}[\phi\_n,U\_n]).
* Assembled a **pivot-weighted Yang–Mills action** on the ladder:  
  [  
  S^{\text{YM}} = \sum\_n g\_n S\_n^{\text{YM}},\quad  
  S^{\text{g+m}} = \sum\_n g\_n S\_n^{\text{g+m}}.  
  ]
* Emphasized the role of the hinge band (n=0) as an **unweighted, 2D area-law pivot** whose Yang–Mills sector is naturally thickened to 4D, while inner/outer bands provide structured corrections.

In the next subsection (9.5), we will turn to the **matter spectrum**, interpreting bandwise eigenstructures of the ladder and gauge sectors as encoding mass hierarchies, mixing patterns, and other features of an abstract matter content, without fixing any empirical values.

**9.5 Context Eigenstates, Projectors & Matter Spectrum**

We now describe how **matter “species”** arise as **eigenstructures** of the ladder + gauge system, without fixing any empirical content. The idea is:

* Combine the **bandwise boundary state spaces** with the **gauge and matter fields**.
* Define suitable **self-adjoint operators** (built from gauge-covariant Laplacians and reproduction kernels).
* Use their **eigenvectors and eigenvalues** to define matter “types” and “mass-like” spectra.

This section is purely structural: it lays out how such a spectrum is defined in principle, not what its numerical values are.

**9.5.1 Combined ladder–gauge–matter state space**

For each band (n):

* Boundary state space: (\mathcal{H}\_n^\partial) (functions/configurations on (\mathcal{G}\_n)).
* Gauge group: (G) (e.g. SU(2), SU(3), or a product), with representation (R) on (V\_R).
* Node matter fields: (\phi\_n: V\_n \to V\_R).

We package band-(n) **matter+gauge states** into an abstract space (\mathcal{H}\_n^{\text{matter}}), which you can think of as:

[  
\mathcal{H}\_n^{\text{matter}}  
\subseteq \mathcal{H}\_n^\partial \otimes \mathcal{F}\_n,  
]  
where:

* (\mathcal{H}\_n^\partial) encodes spatial/boundary dependence on (\mathcal{G}\_n),
* (\mathcal{F}\_n) is a Fock-like or configuration space for fields (\phi\_n) and gauge links (U\_n) at that band.

The **full ladder–gauge–matter space** is then  
[  
\mathcal{H}*{\text{ladder+gauge+matter}}  
:= \bigoplus*{n\in\mathbb{Z}} \mathcal{H}\_n^{\text{matter}}.  
]

This is the space on which we will define operators whose eigenstructures encode the abstract matter spectrum.

**9.5.2 Gauge-covariant Laplacian and band Hamiltonians**

At each band (n), we can define a **gauge-covariant Laplacian** acting on node fields (\phi\_n), using the discrete covariant differences.

Let (w\_n(v\to w)) be positive edge weights, and define the covariant difference  
[  
(\nabla\_n \phi\_n)(v\to w) = U\_n(v\to w),\phi\_n(v) - \phi\_n(w).  
]

A natural **gauge-covariant Laplacian** on (\mathcal{G}\_n) (acting on node fields in representation (R)) is

[  
(\Delta\_n \phi\_n)(v)  
:= \sum\_{w : (v\to w)\in E\_n^{\text{or}}}  
w\_n(v\to w),\bigl( \phi\_n(v) - U\_n(v\to w)^{-1},\phi\_n(w) \bigr).  
]

Alternative forms are possible; the key properties are:

* (\Delta\_n) is **self-adjoint** with respect to a suitable inner product on (\mathcal{H}\_n^\partial) (with node weights (\mu\_n(v))),
* (\Delta\_n) is **gauge covariant**, and the Laplacian acting on gauge-invariant combinations is **gauge invariant**.

The band-(n) **matter Hamiltonian** can then be built from (\Delta\_n) and mass-like terms:

[  
\hat{H}\_n^{\text{matter}}  
:= \frac{1}{2},\bigl( -\Delta\_n + M\_n^2 \bigr),  
]  
where (M\_n^2) is a positive, self-adjoint “mass-squared” operator in (\mathcal{H}\_n^\partial\otimes V\_R) (often taken to be a scalar multiple of the identity in simple models).

The gauge sector has its own band Hamiltonian (\hat{H}\_n^{\text{gauge}}), derived from the Yang–Mills action (S\_n^{\text{YM}}) (e.g. from electric+magnetic energy). We combine them schematically as:

[  
\hat{H}\_n^{\text{g+m}}  
:= \hat{H}\_n^{\text{matter}} + \hat{H}\_n^{\text{gauge}},  
]  
acting on (\mathcal{H}\_n^{\text{matter}}).

**9.5.3 Ladder reproduction kernel and hinge-focused operator**

The ladder also has a **reproduction kernel** (M\_n) on each (\mathcal{H}\_n^\partial), and a pivot-weighted ladder Hamiltonian from Part VII. To define a **global operator** whose eigenstates capture matter types, we combine:

* band Hamiltonians (\hat{H}\_n^{\text{g+m}}),
* reproduction kernels (M\_n),
* and ladder pivot weights (g\_n).

One structurally natural choice is an effective **hinge-focused operator** acting on the full ladder space:

[  
\hat{\mathcal{K}}  
:= \sum\_{n\in\mathbb{Z}} g\_n, \hat{H}*n^{\text{g+m}}  
;+; \sum*{n\in\mathbb{Z}} \lambda\_n \bigl( \mathbf{1} - M\_n \bigr),  
]  
where (\lambda\_n) are non-negative coefficients (e.g. measuring the “cost” of losing memory at band (n)).

Interpretation:

* The (\hat{H}\_n^{\text{g+m}}) terms give the **gauge+matter energy** at each band.
* The ((\mathbf{1}-M\_n)) terms penalize patterns that are not stably reproduced along the ladder.
* The pivot weights (g\_n) emphasize the **hinge and near-hinge bands**, where (g\_n\approx 1), while inner/outer bands may be suppressed or enhanced depending on the form of (g(D(n))).

This (\hat{\mathcal{K}}) is a formal, self-adjoint operator on (\mathcal{H}\_{\text{ladder+gauge+matter}}). Its eigenvalues and eigenvectors will be used to define the matter spectrum.

**9.5.4 Context eigenstates and projectors**

**Definition 9.5.1 (Context eigenstates).**  
A **context eigenstate** (|\Psi\_A\rangle) is an eigenvector of (\hat{\mathcal{K}}):

[  
\hat{\mathcal{K}}|\Psi\_A\rangle = \kappa\_A,|\Psi\_A\rangle,  
]  
with eigenvalue (\kappa\_A \in \mathbb{R}). We assume:

* The set ({|\Psi\_A\rangle}) forms a complete, orthonormal basis (or at least a dense set) in (\mathcal{H}\_{\text{ladder+gauge+matter}}).
* Degeneracies are allowed, so that multiple (|\Psi\_A\rangle) can share the same (\kappa\_A).

Each eigenstate (|\Psi\_A\rangle) decomposes into band components:

[  
|\Psi\_A\rangle = \bigoplus\_{n} |\Psi\_{A,n}\rangle,\quad  
|\Psi\_{A,n}\rangle \in \mathcal{H}\_n^{\text{matter}}.  
]

We define the **projector** onto a given eigenstate as  
[  
\hat{P}*A := |\Psi\_A\rangle\langle\Psi\_A|,  
]  
and, for a degenerate eigenspace (\mathcal{E}*\lambda = \operatorname{span}{|\Psi\_A\rangle : \kappa\_A=\lambda}), the corresponding projector is  
[  
\hat{P}*\lambda := \sum*{A : \kappa\_A=\lambda} |\Psi\_A\rangle\langle\Psi\_A|.  
]

These projectors allow us to define **matter types** as equivalence classes of context eigenstates.

**9.5.5 Matter “species” and mass-like spectra**

We now interpret the eigenvalues (\kappa\_A) (or functions thereof) as **mass-like labels**, and the eigenstates (|\Psi\_A\rangle) as **matter “species”**.

**Definition 9.5.2 (Species and spectrum).**

1. A **species** is an equivalence class of context eigenstates with the same eigenvalue (\kappa\_A) (or, more generally, the same set of invariants, such as gauge representation labels and eigenvalues of (\hat{\mathcal{K}})).
2. The **spectrum** is the set of eigenvalues ({\kappa\_A}), often organized as:
   * a **mass-like spectrum** ({m\_A^2}) obtained from (\kappa\_A) via a monotone function,
   * plus additional discrete labels (e.g. gauge representation indices, internal ladders, etc.).

For example, one can define  
[  
m\_A^2 := f(\kappa\_A),  
]  
where (f) is a strictly increasing function mapping eigenvalues of (\hat{\mathcal{K}}) to positive numbers. The explicit choice of (f) is model-dependent and does not affect the structural fact that the **relative ordering** and **degeneracy pattern** of ({\kappa\_A}) encode a mass-like hierarchy.

Structurally:

* Light species correspond to eigenstates with small (\kappa\_A) (or small (m\_A^2)), often dominated by hinge and near-hinge bands.
* Heavy species correspond to eigenstates with large (\kappa\_A) (or large (m\_A^2)), often involving stronger contributions from inner bands (higher (D(n))) or more complex gauge/Laplacian structure.
* Degenerate eigenvalues correspond to **multiplets**: sets of species that share the same mass-like label but differ in other quantum numbers (e.g. gauge representation components or internal ladder structure).

No particular values or patterns are assumed; the theory merely sets up the framework.

**9.5.6 Band support and localization of eigenstates**

Context eigenstates (|\Psi\_A\rangle) may have different **band support profiles**:

[  
|\Psi\_A|^2 = \sum\_n |\Psi\_{A,n}|^2,\quad  
|\Psi\_{A,n}|^2 = \langle \Psi\_{A,n}|\Psi\_{A,n}\rangle.  
]

We say:

* A species is **hinge-localized** if most of its norm lies in band 0 and a small set of near-hinge bands:  
  [  
  \sum\_{|n|\le n\_0} |\Psi\_{A,n}|^2 \approx 1 \quad\text{for some small }n\_0.  
  ]
* A species is **inner-enhanced** if it has significant support on inner bands (n\ll 0), and **outer-enhanced** if it has significant support on outer bands (n\gg0).

Such band profiles reflect how much a species “lives” in:

* fine-scale internal structure (inner bands),
* the present environment (hinge band),
* or large-scale ambient structure (outer bands).

From the RG viewpoint, hinge-localized species are the most natural to thicken into 4D fields in the next parts of the theory, while inner/outer-enhanced species might correspond to more exotic or heavier sectors.

**9.5.7 Summary**

Section 9.5 has:

* Defined a **combined ladder–gauge–matter state space** (\mathcal{H}\_{\text{ladder+gauge+matter}}).
* Introduced **gauge-covariant Laplacians** (\Delta\_n) and band Hamiltonians (\hat{H}\_n^{\text{g+m}}).
* Combined these with reproduction kernels and pivot weights into an effective **hinge-focused operator** (\hat{\mathcal{K}}) on the full ladder.
* Defined **context eigenstates** (|\Psi\_A\rangle) via (\hat{\mathcal{K}}|\Psi\_A\rangle=\kappa\_A|\Psi\_A\rangle), and projectors onto eigenspaces.
* Interpreted eigenvalues (\kappa\_A) (or functions thereof) as **mass-like labels**, with eigenstates grouped into **species** and **multiplets**.
* Noted that species can be characterized by their **band support profiles**, reflecting how much they live in inner, hinge, or outer bands.

In the next subsection (9.6), we will briefly discuss **gauge bosons and mixing**, showing how excitations of the non-Abelian connections themselves appear as species in this same spectral framework, and how mixing matrices arise from overlaps between eigenstates in different interaction bases.

**9.6 Gauge Bosons & Mixing**

In the previous subsection, matter “species” were defined as **eigenstructures** of the combined ladder–gauge–matter operator (\hat{\mathcal{K}}). We now do the same for the **gauge fields themselves**:

* Treat fluctuations of the non-Abelian connections (U\_n) as dynamical degrees of freedom.
* Identify **gauge boson modes** as eigenstates of appropriate quadratic operators in those fluctuations.
* Describe **mixing** as non-trivial overlaps between eigenbases associated with different interaction sectors or bases.

Everything here is structural: no numerical masses, mixing angles, or specific identifications (like “photon” or “gluon”) are imposed.

**9.6.1 Small fluctuations of link variables**

Consider a fixed background configuration of link variables (\bar{U}\_n(v\to w)) at band (n). We write small fluctuations as

[  
U\_n(v\to w) = \bar{U}\_n(v\to w),\exp\bigl(i A\_n(v\to w)\bigr),  
]

where:

* (A\_n(v\to w)) takes values in the Lie algebra (\mathfrak{g} = \mathrm{Lie}(G)),
* we assume (|A\_n(v\to w)|) is small, so that we can expand the Yang–Mills action to quadratic order in (A\_n).

In matrix form (e.g. adjoint representation), we may write  
[  
A\_n(v\to w) = \sum\_a A\_n^a(v\to w),T^a,  
]  
with generators (T^a) of (\mathfrak{g}).

Gauge transformations act on these fluctuations (in a suitable gauge) as

[  
A\_n(v\to w) \mapsto \Lambda\_n(v),A\_n(v\to w),\Lambda\_n(v)^{-1} + \dots,  
]

where the ellipsis indicates terms involving derivatives of the gauge transformation; at the linearized level, one can impose a gauge-fixing condition (e.g. lattice analogue of Lorenz gauge) so that physical modes are well-defined.

**9.6.2 Quadratic gauge-boson operator at a band**

Expanding the band-(n) Yang–Mills action (S\_n^{\text{YM}}[U\_n]) to quadratic order in (A\_n) around the background (\bar{U}\_n) yields a **quadratic form** in the Lie-algebra-valued edge fields (A\_n(v\to w)):

[  
S\_{n,\text{quad}}^{\text{YM}}[A\_n]  
= \frac{1}{2} \sum\_{(v\to w),(v'\to w')}  
A\_n(v\to w) ,\mathcal{Q}\_n\bigl((v\to w),(v'\to w')\bigr),A\_n(v'\to w'),  
]

where (\mathcal{Q}\_n) is a band-(n) quadratic operator. In practice:

* (\mathcal{Q}\_n) encodes discrete analogues of derivatives and curvature terms on (\mathcal{G}\_n).
* It acts on the space of Lie-algebra–valued edge functions  
  [  
  \mathcal{A}\_n := { A\_n : E\_n^{\mathrm{or}} \to \mathfrak{g} }.  
  ]

After gauge fixing and restriction to the physical subspace (removing pure-gauge and constraint modes), we obtain an effective **gauge-boson operator** on a reduced space (\widetilde{\mathcal{A}}\_n):

[  
\hat{\mathcal{Q}}\_n : \widetilde{\mathcal{A}}\_n \to \widetilde{\mathcal{A}}\_n,  
]  
self-adjoint with respect to a suitable inner product on (\widetilde{\mathcal{A}}\_n).

**9.6.3 Gauge boson eigenmodes and band spectra**

We then define **gauge boson modes** at band (n) as eigenvectors of (\hat{\mathcal{Q}}\_n):

[  
\hat{\mathcal{Q}}*n |B*{n,\alpha}\rangle = \lambda\_{n,\alpha} |B\_{n,\alpha}\rangle,  
]

where:

* (|B\_{n,\alpha}\rangle) is an eigenmode in (\widetilde{\mathcal{A}}\_n) (an abstract ket labelling a fluctuation pattern across edges and Lie-algebra directions),
* (\lambda\_{n,\alpha}\in \mathbb{R}\_{\ge 0}) is an eigenvalue.

Interpreting (\lambda\_{n,\alpha}) as a **mass-like squared eigenvalue**:

* Modes with (\lambda\_{n,\alpha} \approx 0) correspond to **massless** gauge boson modes (at that band).
* Modes with (\lambda\_{n,\alpha} > 0) correspond to **massive** gauge boson modes.

The full **band-(n) gauge boson spectrum** is given by the set ({\lambda\_{n,\alpha}}), together with internal labels: group index, graph pattern, etc.

Band-dependent pivot weights (g\_n) and ladder interactions (via collapse/expansion operators) will then mix these band-local modes into global ladder eigenmodes, just as for matter fields.

**9.6.4 Ladder-level gauge boson operator and eigenstates**

To define **ladder-level** gauge bosons, we combine the bandwise operators (\hat{\mathcal{Q}}\_n) into a global operator:

[  
\hat{\mathcal{Q}}*{\text{ladder}}  
:= \sum*{n} g\_n, \hat{\mathcal{Q}}\_n + \text{(ladder coupling terms)},  
]

where:

* (\hat{\mathcal{Q}}\_n) acts on (\widetilde{\mathcal{A}}\_n),
* ladder coupling terms connect (\widetilde{\mathcal{A}}*n) to (\widetilde{\mathcal{A}}*{n\pm 1}), e.g. via collapse/expansion analogues applied to edge fields or via band-dependent mixing kernels.

The gauge-boson **ladder eigenstates** (|\mathcal{B}\_A\rangle) are then defined by

[  
\hat{\mathcal{Q}}\_{\text{ladder}} |\mathcal{B}\_A\rangle  
= \Lambda\_A |\mathcal{B}\_A\rangle,  
]

with (\Lambda\_A) real eigenvalues. Each (|\mathcal{B}\_A\rangle) decomposes into band components

[  
|\mathcal{B}*A\rangle = \bigoplus\_n |B*{A,n}\rangle,\quad  
|B\_{A,n}\rangle \in \widetilde{\mathcal{A}}\_n.  
]

As in the matter case:

* Hinged–localized gauge bosons: (|B\_{A,0}\rangle) and a few near-hinge components dominate the norm.
* Inner- or outer-enhanced gauge bosons: significant support on (n\ll 0) or (n\gg 0), respectively.

The eigenvalues (\Lambda\_A) provide **mass-like labels** for the ladder gauge bosons, analogous to how (\kappa\_A) did for matter species.

**9.6.5 Mixing between gauge boson sectors**

In general, the gauge sector may contain multiple groups, representations, or channels (e.g. different SU(N) factors, different band ranges). These give rise to **different interaction bases** for gauge bosons:

* A **group basis** (labelled by Lie-algebra generators (T^a)).
* A **band basis** (labelled by (n)).
* A **mass (eigen) basis** (labelled by (A), eigenstates of (\hat{\mathcal{Q}}\_{\text{ladder}})).

Mixing occurs when these bases do not coincide. The transformation between two bases is represented by a **mixing matrix**.

**(a) Group ⇄ mass mixing**

Suppose {|T^a\rangle} labels gauge boson modes associated with individual generators in a given band (or set of bands), and {|\mathcal{B}*A\rangle} is the eigenbasis of (\hat{\mathcal{Q}}*{\text{ladder}}). Then we can write:

[  
|\mathcal{B}*A\rangle = \sum\_a U*{aA},|T^a\rangle,  
]

where (U) is a unitary matrix (or block-unitary, if bands are included):

* Rows labelled by group indices (a),
* Columns labelled by mass eigenmodes (A).

The matrix (U) is the **gauge boson mixing matrix** for that sector. Its entries reflect:

* how each mass eigenstate is composed from underlying group generators,
* how symmetry-breaking and ladder couplings have rotated the basis.

**(b) Band ⇄ mass mixing**

Similarly, in a band basis {|B\_{n,\alpha}\rangle} (band-local modes at band (n)), and a ladder mass basis {| \mathcal{B}\_A\rangle}, we have:

[  
|\mathcal{B}*A\rangle  
= \sum*{n,\alpha} V\_{(n,\alpha)A},|B\_{n,\alpha}\rangle,  
]

with a mixing matrix (V) whose coefficients describe **how much each ladder gauge boson lives on each band** and in which local mode.

Again:

* Band-localized gauge bosons: mixing mostly within a small band range.
* Strongly mixed modes: significant contributions from many bands.

These matrices are structural: they encode overlap of eigenbases of different operators (e.g. ladder Hamiltonian, symmetry generators), not any specific numerical pattern.

**9.6.6 Mixing in the presence of matter**

When matter fields and gauge fields are both present, **interaction bases** are defined by:

* the couplings between matter species (eigenstates of (\hat{\mathcal{K}})) and gauge bosons (eigenstates of (\hat{\mathcal{Q}}\_{\text{ladder}})),
* the representation content (which species transform under which group factors), and
* the ladder band profiles (which bands are relevant for strong couplings).

In such a setting:

* One basis (the “interaction basis”) is convenient for writing gauge–matter couplings (e.g. which species carry which group charges).
* Another basis (the “mass basis”) diagonalizes the matter operator (\hat{\mathcal{K}}) and the gauge-boson operator (\hat{\mathcal{Q}}\_{\text{ladder}}).
* **Mixing matrices** arise as the transformations between these bases, often appearing explicitly in effective couplings and transition amplitudes.

Structurally, these mixing matrices inherit:

* band dependence (due to ladder structure),
* group dependence (due to gauge groups and representations),
* and hinge emphasis (since hinge-localized species and gauge bosons are the ones most directly thickened into 4D effective fields).

The AR formalism does not fix these matrices; it only provides the framework in which they are well-defined and constrained by gauge invariance, ladder symmetries, and the pivot weighting.

**9.6.7 Summary**

Section 9.6 has:

* Treated small fluctuations of the non-Abelian link variables as **gauge-boson degrees of freedom** and, via quadratic expansion of the Yang–Mills action, defined bandwise **gauge-boson operators** (\hat{\mathcal{Q}}\_n).
* Combined these into a **ladder-level operator** (\hat{\mathcal{Q}}\_{\text{ladder}}), whose eigenstates (|\mathcal{B}\_A\rangle) are ladder gauge bosons with mass-like eigenvalues (\Lambda\_A).
* Described **mixing** as the mismatch between:
  + group or band-local bases, and
  + mass-eigenstate bases, with unitary mixing matrices connecting them.
* Noted that, in the presence of matter, the same formalism describes mixing between interaction bases and mass bases, with mixing matrices structurally constrained by ladder, gauge, and pivot structures.

With this, Part 9 (Gauge Structure & Matter Spectrum) is complete at the structural level. The theory now has a fully specified framework for U(1) and non-Abelian gauge fields, ladder-weighted actions, matter spectra, and gauge boson mixing, all built on the context ladder and hinge pivots. Subsequent parts of the unified monograph (Part 10 and beyond) will use this machinery to construct the gravitational sector, nested-time matter, and the qualia–gravity link in the V1 model.

**10. Gravitational Sector & Nested-Time Matter**

**10.1 Gravitational Field Equations from Hinge & Pivot**

In the V1 framework, **gravity** is not an extra field bolted onto an existing spacetime; it is a structural consequence of:

* the **hinge band** (n=0) with (D(0)=2) (area-law boundary),
* the **dimension profile** (D(r)) / (D(n)) and pivot function (g(D)),
* and the **present-act + ladder dynamics** encoded in the master action.

This section sets up the **gravitational field equations** at a purely theoretical level, without numerical constants or empirical identifications. Later sections (10.2–10.3) refine this with scale-dependent deviations and horizon structure.

**10.1.1 Hinge boundary and gravitational potential**

We start from the **hinge boundary** at band (n=0):

* Boundary graph (\mathcal{G}\_0 = (V\_0,E\_0)) with effective 2D dimension (D(0)=2) and area-law scaling.
* Hinge spatial scale (L\_0 = \ell\_{\mathrm{UGM}}).
* Hinge temporal scale (\Theta\_0 = T^\*).
* Conversion constant (c = \ell\_{\mathrm{UGM}}/T^\*).

We introduce a **scalar gravitational potential** as a function on the hinge boundary:

**Definition 10.1.1 (Hinge gravitational potential).**  
A **hinge gravitational potential** is a real-valued function  
[  
\Phi\_0 : V\_0 \to \mathbb{R}  
]  
(or, in a continuum approximation, (\Phi\_0(x)) on a 2D manifold approximating (\mathcal{G}\_0)).

Intuitively:

* (\Phi\_0) measures **deviations** of hinge boundary structure from a reference “flat” configuration.
* Variations of (\Phi\_0) across the hinge encode how context-weighted present-moment configurations are distorted by sources.

To make this precise, we relate (\Phi\_0) to **distortions of the hinge area measure**.

**10.1.2 Area-law distortions and source density**

Let (A\_0(R)) be the hinge boundary measure (Section 6.6), which in the reference case scales as  
[  
A\_0(R) \propto R^2,  
]  
for large graph radius (R). In the presence of sources, we model **local distortions** of this area law via a **surface mass/energy density** (\sigma(x)) on the hinge boundary.

At a structural level:

* (\sigma(x)) represents how much “extra” or “deficit” boundary measure is associated with the presence of nested contexts (matter) at or through (x).
* Positive (\sigma(x)) corresponds to **inward-bending** or “focusing” of boundary structure relative to the reference; negative (\sigma(x)) to **outward-bending** or “dilating.”

We write:

**Definition 10.1.2 (Hinge source density).**  
A **hinge source density** is a function  
[  
\sigma : V\_0 \to \mathbb{R}  
]  
(or (\sigma(x)) in the continuum) that parametrizes deviations from the reference area law.

We now connect (\Phi\_0) and (\sigma) via a hinge-level field equation.

**10.1.3 Hinge Poisson equation (structural form)**

On the hinge boundary we can define a **graph Laplacian** (or continuum Laplacian) (\Delta\_0):

* On the discrete graph (\mathcal{G}*0), (\Delta\_0) acts on functions (f : V\_0\to\mathbb{R}) as  
  [  
  (\Delta\_0 f)(v)  
  := \sum*{w : (v\to w)\in E\_0} a\_0(v,w),\bigl(f(w) - f(v)\bigr),  
  ]  
  with symmetric weights (a\_0(v,w)).
* In a continuum limit, (\Delta\_0) becomes the Laplace–Beltrami operator on a 2D surface approximating the hinge boundary.

We now **postulate** a hinge-level Poisson-type equation:

**Axiom 10.1.3 (Hinge Poisson equation).**  
The hinge gravitational potential (\Phi\_0) and the hinge source density (\sigma) are related by  
[  
\Delta\_0 \Phi\_0(x) = \kappa\_0,\sigma(x),  
]  
for some positive constant (\kappa\_0) (the structural gravitational coupling at the hinge).

Interpretation:

* Positive (\sigma) (mass/energy-like source) induces concavity in (\Phi\_0);
* (\Phi\_0) is defined up to an additive constant;
* The equation is **structural**: it expresses how boundary distortions (encoded in (\Phi\_0)) are tied to sources on the hinge.

This is the **2D pivot equation**. To arrive at a familiar 3D/4D gravity equation, we must:

1. **Thicken** the hinge boundary into a radial shell, and
2. Lift (\Phi\_0) to a scalar potential (\Phi) on a 3D (or 4D) domain.

**10.1.4 Hinge thickening and 3D inverse-square field**

Consider a **radial thickening** of the hinge boundary into a 3D space, with radial coordinate (r) and angular coordinates attached to the hinge boundary. At a structural level:

* The hinge boundary (\mathcal{G}*0) becomes the “surface” at a reference radius (in units of (\ell*{\mathrm{UGM}})).
* The area-law scaling (A\_0(R)\propto R^2) implies that, in the thickened domain, the **flux through a sphere of radius (R)** is naturally distributed over an area (4\pi R^2)-like surface.

We define a **3D gravitational potential** (\Phi(\mathbf{x})) such that:

* On the hinge surface (reference radius), (\Phi(\mathbf{x})) restricts to (\Phi\_0).
* Away from the hinge surface, (\Phi) satisfies a **3D Poisson equation**:  
  [  
  \nabla^2 \Phi(\mathbf{x}) = \kappa,\rho(\mathbf{x}),  
  ]  
  where:
  + (\nabla^2) is the 3D Laplacian,
  + (\rho(\mathbf{x})) is a **volume source density** derived from (\sigma(x)) by appropriate thickening,
  + (\kappa>0) is a structural coupling constant.

In spherical symmetry with a compact source of total “mass” (M):

* (\rho(\mathbf{x}) = 0) outside the source support,
* The field equation reduces to  
  [  
  \nabla^2 \Phi(r) = 0\quad\text{for } r > r\_{\text{source}},  
  ]
* The solution with asymptotically zero potential is  
  [  
  \Phi(r) = - \frac{C M}{r},  
  ]  
  for some constant (C>0).

The corresponding **gravitational field** (\mathbf{g}(\mathbf{x})) is  
[  
\mathbf{g}(\mathbf{x}) := -\nabla \Phi(\mathbf{x}),  
]  
so in spherical symmetry  
[  
|\mathbf{g}(r)| \propto \frac{1}{r^2}.  
]

Thus, the **inverse-square law** emerges structurally from:

* the **area-law nature of the hinge**, and
* the 3D Poisson equation obtained by thickening the hinge boundary.

We emphasize that no numerical values (e.g. Newton’s (G)) are specified; only the **functional form** of the field equation and its solutions are asserted.

**10.1.5 Ladder-level gravitational action and field equations**

The hinge-level equations above must be embedded into the **full ladder action**. To do this, we introduce a **gravitational sector** of the master action, analogous to the gauge sectors of Part 9.

At a schematic level:

* Let (\Phi\_n) be a gravitational potential-like field on (\mathcal{G}\_n) (band-(n) boundary).
* Define bandwise gravitational actions (S\_n^{\text{grav}}[\Phi\_n]), constructed from:
  + bandwise Laplacians (\Delta\_n),
  + bandwise source densities (\sigma\_n),
  + and bandwise pivot weights (g\_n).

A simple structural choice is:

[  
S\_n^{\text{grav}}[\Phi\_n]  
:= \frac{1}{2}\sum\_{v\in V\_n} \mu\_n(v),\Phi\_n(v),(-\Delta\_n \Phi\_n)(v)  
- \sum\_{v\in V\_n} \mu\_n(v),\Phi\_n(v),\sigma\_n(v),  
]  
where (\mu\_n(v)) are node weights. Varying with respect to (\Phi\_n) yields the discrete Poisson equation  
[  
\Delta\_n \Phi\_n(v) = \sigma\_n(v).  
]

The **total gravitational action** on the ladder is then:

[  
S^{\text{grav}}[{\Phi\_n}*n]  
:= \sum*{n} g\_n, S\_n^{\text{grav}}[\Phi\_n],  
]  
with **pivot weights** (g\_n = g(D(n))). At the hinge band (n=0):

* (g\_0 = 1),
* (S\_0^{\text{grav}}) reduces to the hinge Poisson structure,
* thickening and division-by-zero promote (S\_0^{\text{grav}}) to a 4D gravitational field action.

The **field equations** for gravity in the ladder are then obtained by varying (S^{\text{grav}}) with respect to (\Phi\_n), leading to:

[  
g\_n,\Delta\_n \Phi\_n(v) = g\_n,\sigma\_n(v) + \text{(ladder coupling terms)},  
]  
where the ladder coupling terms come from cross-band dependencies (e.g. through collapse/expansion operators or constraints linking (\Phi\_n) across (n)).

At the hinge, these reduce to the pivot equation (\Delta\_0 \Phi\_0 = \kappa\_0 \sigma\_0), which, after thickening, yields the familiar 3D Poisson equation for a scalar gravitational potential.

**10.1.6 Interpretation and scope**

The construction above establishes:

1. A **hinge-level gravitational potential** (\Phi\_0) defined on a 2D area-law boundary, sourced by a surface density (\sigma).
2. A **hinge Poisson equation** (\Delta\_0 \Phi\_0 = \kappa\_0 \sigma) as a pivot relation between boundary distortions and sources.
3. A **thickening procedure** that lifts (\Phi\_0) to a 3D potential (\Phi(\mathbf{x})) obeying a Poisson equation (\nabla^2\Phi = \kappa\rho), with **inverse-square behavior** in spherical symmetry.
4. A **ladder-level gravitational action** where each band contributes a Poisson-type term weighted by (g\_n), with the hinge band unweighted and structurally singled out.

This is not yet **full general relativity**; it is the **scalar, Newtonian-level end** of the gravitational sector in AR. In later parts (and in the higher-level unified 4D equation), additional structure (including curvature of null cones, collapse kernels, and the influence of context dynamics on effective metrics) will be introduced to capture richer gravitational phenomena.

For the purposes of this core V1 volume, however, 10.1 establishes the **gravitational field equations** as a direct outgrowth of:

* the hinge area-law nature of the context ladder, and
* the pivot structure encoded by (D(n)) and (g(D)).

**10.2 Scale-Dependent Deviations from the Inverse-Square Law**

In Section 10.1 we treated gravity at the **hinge** as a scalar potential obeying a Poisson-type equation, whose thickening yields a 3D inverse-square field in the weak-field, spherically symmetric limit. That description is **pivot-local**: it assumes that the hinge band is effectively isolated and that the dimension profile (D(n)) and pivot weights (g\_n) do not vary significantly over the scales of interest.

In the full ladder, however:

* The **dimension curve** (D(n)) differs from 2 away from the hinge,
* The **pivot weights** (g\_n = g(D(n))) vary across bands,
* And bands can couple through the collapse/expansion algebra and reproduction kernels.

These features naturally introduce **scale-dependent deviations** from the pure (1/r^2) behavior. This section describes that structure in a purely theoretical way, without assigning any numerical values or empirical interpretations.

**10.2.1 Hinge approximation vs full ladder**

The pure inverse-square law arises when we make three approximations:

1. **Hinge-band dominance**  
   Only the hinge band (n=0) contributes appreciably to the gravitational sector of the action:  
   [  
   S^{\text{grav}}[{\Phi\_n}]  
   \approx S\_0^{\text{grav}}[\Phi\_0],  
   ]  
   with all (g\_{n\neq 0} S\_n^{\text{grav}}) treated as negligible corrections.
2. **Flat hinge boundary**  
   The hinge boundary (\mathcal{G}\_0) is approximated by a flat or weakly perturbed 2D geometry, so that the Laplacian (\Delta\_0) can be thickened to the standard 3D Laplacian (\nabla^2) in a straightforward way.
3. **Single-scale regime**  
   The radial range of interest lies within a window where the **effective scale** corresponds to a narrow band of (n) (or (r)) around the hinge, so that (D(n)\approx 2) and (g\_n\approx 1).

Under these assumptions, we get:

* a 3D Poisson equation (\nabla^2\Phi = \kappa\rho),
* with weak-field solutions (\Phi(r)\propto -1/r),
* and (|\mathbf{g}(r)| \propto 1/r^2).

When these approximations are relaxed, the full ladder structure modifies the effective gravitational response.

**10.2.2 Dimension profile and effective radial measure**

The **dimension profile** (D(r)) (or its discrete version (D(n))) controls how “boundary measure” scales with radius:

* At the hinge: (D(0)=2) → area-like scaling.
* Inner bands ((r<0), (n<0)): (D>2) → more volume-like.
* Outer bands ((r>0), (n>0)): (D<2) → more filament-like.

When we thicken the hinge boundary into a radial shell, the effective “surface measure” at radius (R) can be modeled as

[  
\mathcal{A}*{\text{eff}}(R)  
\propto R^{D*{\text{eff}}(R)},  
]  
where (D\_{\text{eff}}(R)) is a **radius-dependent effective dimension** induced by the ladder. At the hinge, (D\_{\text{eff}}(R) \approx 2). Away from the hinge, (D\_{\text{eff}}(R)) drifts toward the inner or outer limits of the dimension profile.

For a spherically averaged field, the gravitational flux through a “surface” of radius (R) is distributed over (\mathcal{A}\_{\text{eff}}(R)), so the **field magnitude** scales structurally like

[  
|\mathbf{g}(R)| \propto \frac{1}{\mathcal{A}*{\text{eff}}(R)}  
\propto R^{-D*{\text{eff}}(R)}.  
]

In the hinge-dominated regime, (D\_{\text{eff}}(R)\approx 2), so (|\mathbf{g}(R)|\propto 1/R^2). In inner- or outer-dominated regimes, deviations in (D\_{\text{eff}}(R)) lead to power-law departures from (1/R^2).

We emphasize:

* This is a **structural statement**: it does not assign a specific function to (D\_{\text{eff}}(R)); it only notes that any ladder-induced variation of effective dimensionality shows up as a scale-dependent power of (R) in the field’s falloff.

**10.2.3 Pivot weights and band contributions**

The **pivot weights** (g\_n = g(D(n))) enter the gravitational action as band-dependent multipliers. In the schematic bandwise action

[  
S^{\text{grav}}[{\Phi\_n}]  
= \sum\_{n} g\_n, S\_n^{\text{grav}}[\Phi\_n],  
]  
each (S\_n^{\text{grav}}) contains terms like

[  
S\_n^{\text{grav}}[\Phi\_n]  
\sim \frac{1}{2}\Phi\_n(-\Delta\_n \Phi\_n) - \Phi\_n \sigma\_n,  
]  
so the **effective strength** of the band-(n) contribution is modulated by (g\_n).

If we expand the full gravitational potential as a superposition of band contributions,

[  
\Phi(\mathbf{x}) \sim \sum\_n \Phi\_{(n)}(\mathbf{x}),  
]  
with each (\Phi\_{(n)}) associated with a band-(n) field (\Phi\_n) and weight (g\_n), then:

* Bands with (g\_n \ll 1) contribute weakly,
* Bands with (g\_n \approx 1) (near the hinge) dominate,
* Bands with larger or smaller weights (depending on the shape of (g(D))) can enhance or suppress the response in certain radial ranges.

Thus, **scale-dependent deviations** arise not only from the geometric dimension profile but also from the way pivot weights distribute “responsiveness” across bands.

**10.2.4 Effective radial equation with ladder corrections**

In a coarse-grained, spherically symmetric, weak-field regime, we can summarize the impact of ladder corrections by an **effective radial equation** of the form

[  
\frac{1}{R^{D\_{\text{eff}}(R)-1}}  
\frac{\mathrm{d}}{\mathrm{d}R}  
\left( R^{D\_{\text{eff}}(R)-1} \frac{\mathrm{d}\Phi}{\mathrm{d}R} \right)  
= \kappa\_{\text{eff}}(R),\rho\_{\text{eff}}(R),  
]  
where:

* (D\_{\text{eff}}(R)) encodes the ladder-induced effective dimension at radius (R),
* (\kappa\_{\text{eff}}(R)) is an effective gravitational coupling, influenced by the bandwise pivot weights and their RG flow,
* (\rho\_{\text{eff}}(R)) is an effective source profile after integrating out ladder structure at smaller and larger bands.

In the hinge-limit:

* (D\_{\text{eff}}(R)\to 2),
* (\kappa\_{\text{eff}}(R)) approaches a constant (\kappa),
* (\rho\_{\text{eff}}(R)) reduces to the usual volume source density,

and the equation reduces to the standard 3D Poisson equation:  
[  
\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}  
\left( R^2 \frac{\mathrm{d}\Phi}{\mathrm{d}R} \right)  
= \kappa \rho(R).  
]

Away from this regime, deviations in (D\_{\text{eff}}(R)) and (\kappa\_{\text{eff}}(R)) lead to modified field profiles, which remain fully encoded in the ladder structure.

**10.2.5 Ladder RG view: running gravitational response**

From the RG perspective of Section 7.6, the **effective gravitational response** at scale (R) is obtained by:

* integrating out inner bands (high-context “UV”) and outer bands (low-context “IR”),
* rescaling the ladder coordinate and fields,
* and reading off the effective couplings at the scale corresponding to (R).

The result is a **running gravitational coupling** (\kappa\_{\text{eff}}(R)) and a running effective dimension (D\_{\text{eff}}(R)). Structurally:

* Inner bands ((n\ll 0)): influence the **high-acceleration / small-scale** behavior.
* Outer bands ((n\gg 0)): influence the **low-acceleration / large-scale** behavior.
* The hinge and near-hinge bands: control the **intermediate, “Newtonian” regime** where the inverse-square law is a good approximation.

The ladder RG therefore gives a precise meaning to the idea that gravity’s effective behavior **flows with scale**. This is captured in the ladder by the evolution of:

* (D(n)) and (g(D(n))),
* bandwise gravitational couplings and source definitions,
* and the reproduction kernels that encode how patterns propagate across bands.

**10.2.6 Summary**

Section 10.2 has:

* Highlighted that the pure (1/r^2) inverse-square law is a **hinge approximation**, valid when:
  + the hinge band dominates,
  + the hinge boundary is approximately flat,
  + and the radial range stays near (D\_{\text{eff}}(R)\approx 2) and (g\_n\approx 1).
* Explained how the **dimension profile** (D(r))/(D(n)) produces an effective running dimension (D\_{\text{eff}}(R)) that controls how “surface measure” scales with radius and thus how (|\mathbf{g}(R)|) deviates from (\propto 1/R^2).
* Shown that **pivot weights** (g\_n) modulate band contributions to the gravitational action, leading to additional scale-dependent corrections.
* Introduced a schematic **effective radial equation** with (D\_{\text{eff}}(R)) and (\kappa\_{\text{eff}}(R)), from which modified field profiles follow.
* Interpreted these features via the ladder **RG flow**, where inner/outer bands encode UV/IR corrections to the hinge-level gravitational response.

In the next subsection (10.3), we will turn to **horizons and entropy in ladder language**, showing how the combination of area-law hinge behavior and ladder dynamics naturally yields horizon formation, entropy scaling with area, and structural log corrections in the V1 framework.

**10.3 Horizons & Entropy in Ladder Language**

In the V1 framework, **horizons** are not introduced as separate geometric objects; they are **emergent boundaries of the context ladder** where:

* the **present-act + ladder dynamics** prevent certain information from returning to the hinge context, and
* the **ON/IN ledger** plus area-law hinge structure naturally give an **entropy proportional to boundary “area”**, with controlled corrections.

This section gives a structural account of:

* what a horizon is in ladder terms,
* how horizon entropy arises from ON-state counting at a pivot band, and
* why the leading term is area-like, with subleading logarithmic corrections.

No numerical constants, temperatures, or empirical identifications are introduced.

**10.3.1 Structural notion of a horizon**

At the hinge (and its thickened radial neighborhood), the ladder provides:

* a radial coordinate (r) or band index (n),
* a family of **Collective Spheres** (\mathrm{CS}\_n) at different context levels,
* a dimension profile (D(r)) or (D(n)), and
* a gravitational potential (\Phi) and field (\mathbf{g}) derived from the hinge Poisson structure.

A **horizon** is then characterized structurally as a context-surface beyond which:

* **return of ON-states to the hinge band is suppressed or blocked** by the ladder dynamics, and
* **collapse/expansion operators** plus reproduction kernels effectively decouple inner bands (deeper than the horizon) from the portion of the hinge reachable by outward paths.

We capture this by the idea of a **trapped region** in the ladder:

A region of bands and configurations is **trapped** relative to the hinge if, once carriers enter that region, no admissible sequence of flip + collapse/expansion operations brings them back to hinge-level carriers with non-negligible weight.

The **horizon** is then the **boundary band/surface** of such a trapped region, viewed from the hinge.

In spherical symmetry, this manifests as a radial radius (R\_H) (measured in hinge units) such that:

* carriers starting inside (R\_H) cannot, in practice, generate hinge-level carriers outside (R\_H) with appreciable probability,
* while carriers starting outside (R\_H) can still propagate and interact in the outer ladder.

**10.3.2 Hinge band and trapped CS structure**

At the hinge, consider a family of CSs that describe “shells” at increasing radial bands or radii. Let:

* (\mathrm{CS}\_0) be a hinge CS at the reference radius,
* (\mathrm{CS}\_{\text{inner}}) be CSs nested inside a candidate horizon radius,
* (\mathrm{CS}\_{\text{outer}}) be CSs outside it.

We say that a **horizon CS** (\mathrm{CS}\_H) is a boundary CS such that:

1. **Inside trapping**  
   Once carriers are associated with inner CSs whose boundary lies at or inside (\mathrm{CS}\_H), the ladder dynamics (collapse/expansion + gravitational sector) drive them to a regime where:
   * reproduction kernels (M\_n) favor inward flow,
   * band couplings suppress outward return to hinge band states outside (R\_H).
2. **Outside accessibility**  
   Carriers associated with CSs outside (\mathrm{CS}\_H) still have access to the hinge and external bands; they form part of the “outside world.”

Operationally:

* The **horizon** is the last band/CS at which outward-directed ON-states are still meaningfully available to the hinge.
* Beyond it, inward-directed ON-states accumulate in a trapped region, which we can think of as a **collapsed interior**.

**10.3.3 Entropy as ON-state counting at the horizon**

Given a horizon CS (\mathrm{CS}\_H), we define **horizon entropy** in terms of ON-state counting consistent with:

* a fixed macroscopic horizon configuration (e.g., fixed total “mass/energy” and radius), and
* the ledger constraints (capacity, record vs exposure) on the CS.

Let:

* (X\_H) be the set of **microscopic ON-configurations** on and just inside (\mathrm{CS}\_H) that are compatible with a given macroscopic horizon specification,
* each configuration (x\in X\_H) correspond to a distinct pattern of ON → IN potentials that, from the outside, look the same.

**Definition 10.3.1 (Horizon entropy).**  
The **horizon entropy** (S\_H) is defined (up to an overall constant) as a logarithmic measure of the number of compatible ON-configurations:

[  
S\_H ;\propto; \log \bigl( # X\_H \bigr),  
]

or, in a more refined setting, as a suitable entropy functional over a probability distribution on (X\_H) induced by the ladder dynamics.

In other words:

* The entropy counts, in a coarse-grained sense, the number of **hidden ON-states** that correspond to the same macroscopic horizon boundary as seen from the hinge and outer bands.
* The ledger semantics (record vs exposure, capacity) ensures that these are truly **unresolved degrees of freedom** from the outside viewpoint: they are potentials that cannot return to external CSs once they cross the horizon.

**10.3.4 Area law and logarithmic corrections**

Because the horizon is realized as a boundary at or near the **hinge band**:

* Its **effective dimension** is close to 2 (area-law scaling).
* The number of independent ON-configurations grows roughly exponentially with the number of boundary “patches” (nodes) at the horizon, which is proportional to the horizon **area** in hinge units.

Let:

* (A\_H) be the horizon’s effective area measure (e.g., number of boundary nodes times a node-area weight).

Then structurally:

1. **Leading area law**  
   The leading behavior of the entropy is  
   [  
   S\_H \sim \alpha, A\_H,  
   ]  
   for some positive constant (\alpha) (an entropy-per-unit-area factor). This comes from the exponential growth of ON-configurations with boundary patches: (# X\_H \sim \exp(\alpha A\_H)).
2. **Subleading logarithmic corrections**  
   The **memory dimension** (D\_{\mathrm{mem}}(n)), the reproduction kernels near the horizon, and the finite ladder thickness combine to generate **subleading corrections** proportional to (\log A\_H):

[  
S\_H \sim \alpha, A\_H + \beta, \log A\_H + \dots,  
]  
where (\beta) is a model-dependent constant and “(\dots)” indicates further suppressed terms (e.g. inverse powers of (A\_H)).

Intuitively:

* + The leading term counts “independent bits per boundary patch.”
  + The logarithmic term arises from **correlations and constraints** in how degrees of freedom can be distributed near the horizon, as encoded in the reproduction kernels and ladder couplings.

No specific coefficients (\alpha, \beta) are fixed in this volume; only the **functional dependence on area and its logarithm** is asserted structurally.

**10.3.5 Thickening to 4D and horizon thermodynamics (structural)**

When we apply the **division-by-zero / hinge-thickening** procedure to the gravitational sector:

* The hinge horizon CS (\mathrm{CS}\_H) is promoted to a 3D (spatial) horizon surface embedded in a 4D manifold (M).
* The horizon entropy (S\_H) is attached to this 3D surface via the same area/ON-counting arguments, now interpreted in 4D geometry.
* The ladder dynamics (including present-act evolution and gravitational sector) induce relations between:
  + changes in horizon area/entropy,
  + energy flux across the horizon,
  + and effective “surface gravity” associated with the horizon’s gravitational field.

Structurally, these are the ingredients for **horizon thermodynamics**:

* A relation of the form  
  [  
  \delta S\_H \propto \delta A\_H,  
  ]  
  combined with energy flux constraints from the gravitational field equations, yields analogues of **first-law-type relations** for horizons (change in horizon entropy related to energy flux and potential-like parameters).

The V1 volume does not develop full thermodynamics; it only:

* identifies the **origin** of area-law entropy and log corrections,
* and notes that the combination of hinge area-law structure, ON/IN ledger, and ladder dynamics is sufficient to recover the qualitative ingredients of horizon thermodynamics when thickened to 4D.

**10.3.6 Summary**

Section 10.3 has:

* Defined **horizons** as context-ladder boundaries (CSs) beyond which ON-states are trapped, unable to return to the hinge with appreciable weight.
* Tied horizons to **trapped regions** in the ladder, with the horizon CS as the boundary between accessible and inaccessible bands from the hinge viewpoint.
* Defined **horizon entropy** (S\_H) as a logarithmic measure of horizon-compatible ON-configurations hidden behind the horizon, subject to ledger constraints.
* Shown structurally why horizon entropy obeys a leading **area law** (S\_H\sim \alpha A\_H) with subleading **logarithmic corrections** (S\_H\sim \alpha A\_H + \beta\log A\_H + \dots).
* Explained how hinge thickening to 4D and the gravitational sector’s dynamics provide the ingredients for horizon thermodynamics, without fixing any numerical constants or physical identifications.

In the next subsection (10.4), we turn to **nested-time matter across scales**, formalizing how the same context ladder that supports gravity and horizons also encodes inner (e.g. neural/biological) and outer (e.g. planetary/cosmic) structures as “nested-time matter” within the AR framework.

**10.4 Nested-Time Matter Across Scales**

The same **context ladder** that supports the gravitational sector also organizes what we ordinarily call “matter” across scales. In the V1 formalism, matter is not a separate substrate added into a pre-existing spacetime; rather, it is:

Stable, nested patterns of **PMS/CS configurations** distributed over inner and outer bands of the ladder, persisting across context-time.

This section defines **nested-time matter** and how it appears at different scale ranges (inner, hinge, outer), entirely in terms of ladder objects.

**10.4.1 Nested-time systems**

We begin by defining a **nested-time system**: a coherent configuration on the ladder that has an identifiable “center” and a structured set of inner and outer contexts.

**Definition 10.4.1 (Nested-time system).**  
A **nested-time system** is specified by:

1. A distinguished **center band** (often but not necessarily ).
2. A family of **Collective Spheres** (CSs)

indexed by a contiguous band interval , such that:

* + for each , is a CS at band ,
  + is the **center CS**,
  + for , represent **inner contexts** (sub-structure),
  + for , represent **outer contexts** (environment).

1. A **compatibility structure**: collapse/expansion and reproduction operations map between the in a way that preserves a notion of “same system” across bands.

Intuitively:

* is a **stack** of CSs over bands, all interpreted as “the same object” viewed at different contextual depths (inner) and extents (outer).
* The context ladder organizes these views from finer internal structure to coarser environmental embedding.

**10.4.2 Nested-time matter as stable ladder patterns**

We now define **nested-time matter** as nested-time systems that are dynamically stable over context-time.

**Definition 10.4.2 (Nested-time matter).**  
A nested-time system is called **nested-time matter** if:

1. **Context-time persistence**  
   There exists a context-time interval such that for each , there is a family

satisfying Definition 10.4.1 and related by admissible present-act dynamics.

1. **Reproduction stability across bands**  
   For each band , the bandwise reproduction kernel has at least one long-lived mode (Section 6.3) that supports across . That is, the system’s band- boundary state lies in (or near) the subspace of eigenmodes of with eigenvalues close to 1.
2. **Coherent center**  
   The center CS maintains a consistent identity across , in the sense that:
   * it remains within a small neighborhood of a stable orbit in the band- boundary state space,
   * inner and outer CSs remain consistently nested around it.

Nested-time matter is therefore:

* **extended in band space** (inner and outer CSs),
* **persistent in context-time** (reproduction stability),
* and **centered** (a distinguishable nested core).

This is the ladder-level analogue of an object with internal structure and external environment that persists over time.

**10.4.3 Inner, hinge, and outer nested-time sectors**

A given nested-time system may span many bands. Structurally, we can distinguish three nested sectors:

1. **Inner nested-time matter (sub-structure)**
   * Bands .
   * Higher IN dimension (more volume-like).
   * Smaller spatial scales ; shorter temporal scales .
   * Role: encode **internal degrees of freedom** (subcomponents, fine-grained states) that collectively support the center CS.
2. **Center band / hinge-level nested matter**
   * Band (often chosen as the hinge in many models).
   * Effective 2D boundary dimension if ; more generally, the band at which we define the **primary present environment** for the system.
   * Role: the **reference layer** where we describe the system’s present state and its coupling to gravity and gauge sectors.
3. **Outer nested-time matter (environment)**
   * Bands .
   * Lower IN dimension (more filament-like).
   * Larger spatial scales ; longer temporal scales .
   * Role: encode the **larger structures** within which the system is embedded (local neighborhoods, planetary, astrophysical, or cosmic environments in symbolic terms).

In this way, the **same nested-time system** is seen:

* as a **collection of parts** when we look inward (inner bands),
* as an **object** when we look at the center band,
* as an **element of a larger structure** when we look outward (outer bands).

**10.4.4 Nested-time worldlines**

We now define a **worldline** for nested-time matter in context-time and band space.

**Definition 10.4.3 (Nested-time worldline).**  
A **nested-time worldline** of a system is a map

such that:

1. For each , the family satisfies nestedness and center conditions (Definitions 10.4.1–10.4.2).
2. The evolution is generated by admissible present-act dynamics and ladder flows (collapse/expansion + reproduction).
3. The center band CS follows a path in its band-level configuration space that corresponds, when thickened, to a trajectory in an emergent 4D spacetime.

Thus:

* The **ladder-level worldline** is a history of nested CSs across bands and context-time.
* The **emergent 4D worldline** is a projection of obtained by:
  + selecting the center band,
  + applying the hinge-thickening and division-by-zero mapping,
  + and reading off the effective 4D trajectory of the system’s center.

Nested-time matter therefore has a **multi-band worldline** whose projection yields the familiar trajectory of an object.

**10.4.5 Interaction between nested-time matter and gravity (structural view)**

In the gravitational sector, nested-time matter acts as a source via its **bandwise presence**:

* Inner nested-time structure modifies **local hinge boundary conditions** and thus contributes to the hinge source density .
* Outer nested-time structure shapes **background** and **context couplings** that affect the center’s motion (e.g., trajectories of its worldline).

Structurally:

1. **Source mapping**  
   A nested-time system induces a collection of bandwise source densities . The **effective gravitational source** seen at the center band is obtained by combining these via ladder couplings and pivot weights:

where represent ladder transport operations (e.g. collapse/expansion-induced mappings).

1. **Response in**   
   The hinge/center potential then satisfies a Poisson-type equation with source , as in Section 10.1, leading to a gravitational field that influences the center CS trajectory.
2. **Back-action**  
   The worldline feeds back into the ladder via changes in:
   * the distribution of nested CSs,
   * band support of the system,
   * and, over longer scales, the dimension and coupling profiles.

This establishes, at a conceptual level, how **nested-time matter and gravity** mutually influence one another in the AR ladder, without invoking a separate “matter sector” defined on a fixed background.

**10.4.6 Summary**

Section 10.4 has:

* Defined **nested-time systems** as stacks of CSs across bands, with a center band and nested inner/outer contexts.
* Defined **nested-time matter** as nested-time systems that are dynamically stable across context-time, with reproduction stability and a coherent center.
* Distinguished **inner**, **center/hinge**, and **outer** nested-time sectors, each corresponding to different bands and scale ranges.
* Introduced **nested-time worldlines**, histories of nested CSs across bands and context-time, whose projections yield emergent 4D trajectories.
* Sketched at a structural level how nested-time matter sources and responds to gravity through bandwise source densities and hinge-level potentials.

In the next subsection (10.5), we will define **qualia** in this framework as equivalence classes of nested ladder configurations centered on a present-moment band, connecting the nested-time matter picture to the theory’s core interpretation of conscious experience.

**10.5 Qualia as Equivalence Classes of Ladder Configurations**

In earlier parts, “qualia” were introduced informally as labels for **present-moment content** carried by PMSs and CSs. Here we give a **purely structural definition**:

A **quale** is an equivalence class of ladder configurations that share the same present-moment structure at a chosen center band, modulo neutral moves, internal refinements, and external embeddings.

This section formalizes that statement within the context-ladder, nested-time, and gravitational structures developed so far.

**10.5.1 Present-centered ladder configurations**

We first define what it means to talk about a **present-centered configuration** of the ladder.

Fix a center band (often , the hinge band), and a context-time . Consider:

* A family of CSs forming a nested-time system (as in 10.4).
* A corresponding family of carriers attached to the CSs, with PMS.
* Bandwise gauge, matter, and gravitational fields associated with these carriers (Sections 9 and 10).

**Definition 10.5.1 (Present-centered ladder configuration).**  
A **present-centered ladder configuration** at is a collection

satisfying:

1. **Nestedness**: form a nested-time system with center band .
2. **Consistency**: the carriers and fields at each band are compatible with the CSs and obey the ladder dynamics (present-act, collapse/expansion, reproduction, etc.).
3. **Present focus**: is singled out as the “present environment” of interest; inner bands are treated as sub-structure, outer bands as embedding.

Thus, encodes **everything structurally relevant** to the present state at band , including inner and outer context structure, at the given context-time .

**10.5.2 Equivalence relation on ladder configurations**

Different ladder configurations may be **indistinguishable** at the center band, even if they differ in sub-structure or distant environment. We now formalize “indistinguishable at the present” as an equivalence relation.

Let be the set of all present-centered ladder configurations at band (for some fixed context-time ; we suppress in notation when not needed).

We define an equivalence relation on generated by the following transformations:

1. **Neutral moves (tick-level and ladder-level)**
   * Tick-level neutral words (Section 2.4) that act as identity on the carriers.
   * Ladder-level neutral composites of collapse/expansion (Section 6.2) that act as identity on band- boundary states.
2. **Inner refinements and coarse-grainings**  
   Operations that:
   * modify inner bands via admissible collapse/expansion and reproduction steps,
   * leave band- carriers and band- observable structures invariant,
   * and preserve the hinge-level Born-style statistics for any CS sampling at band .
3. **Outer embedding changes**  
   Operations that:
   * modify outer bands (e.g. different ambient configurations, distant nested-time matter) via admissible ladder moves,
   * keep the band- carriers and band- local gravitational/gauge/matter observables unchanged,
   * and maintain the same effective hinge gravitational potential in some neighborhood of the center CS.

More precisely:

**Definition 10.5.2 (Present-content equivalence).**  
Two present-centered ladder configurations are **present-content equivalent** at band , written

if can be obtained from by a finite sequence of transformations of types (1)–(3) above.

This relation is:

* **Reflexive:** identity transformation is allowed.
* **Symmetric:** each allowed move has an inverse within the same class.
* **Transitive:** compositions of allowed moves remain allowed.

Thus is an equivalence relation on .

**10.5.3 Qualia as equivalence classes**

We can now define qualia in purely structural terms.

**Definition 10.5.3 (Quale at band ).**  
A **quale** (at center band ) is an equivalence class

of present-centered ladder configurations under .

Interpretation:

* Each equivalence class is the set of all ladder configurations that share the **same present-moment content** at band , up to neutral moves, internal refinements, and changes in distant environment that do not alter band- observables.
* A single ladder configuration is just a **representative** of the class; the quale is the class itself.

Qualia, in this sense, are:

* **Intrinsic** to the ladder structure: they are subsets of .
* **Present-centered:** defined relative to a particular center band .
* **Invariant** under symmetries that leave band- present content unchanged.

We may denote the set of all qualia at band by

**10.5.4 Qualia and CS sampling**

The Born-style CS sampling procedure (Section 7.5) yields present-plane amplitude sets over outcome basins at a given CS and band. For the center CS :

* **Sampling** at maps a ladder state (or configuration) to amplitudes ,
* In decohered regimes, it yields probabilities for outcomes .

Within a given equivalence class :

* All representatives share the **same hinge/center CS sampling statistics** for any admissible sampling setup localized at band :
  + same set of outcome basins ,
  + same amplitude set up to phase conventions,
  + same Born-style probabilities .

Conversely, if two ladder configurations produce **different** CS sampling structures at band (different in an invariant sense), they cannot be equivalent under , and thus represent **different** qualia.

Structurally:

* CS sampling provides an **operational handle** on the equivalence classes: qualia correspond to **distinct CS sampling outcome structures** at the center band, modulo neutral symmetries and internal/external refinements that do not change those structures.
* The qualia space can be thought of as the **space of possible CS sampling structures** at the present environment band.

**10.5.5 Relation to nested-time matter and gravity**

Qualia are not separate from nested-time matter and gravity; they are **specific cuts** through the combined structure:

* A nested-time matter system has a center CS and band components .
* A gravitational configuration provides hinge/center potentials and bandwise fields shaping the nested-time worldline .
* A **present-centered ladder configuration** includes both the nested-time matter and gravitational sectors.

The **quale** is the equivalence class of all such configurations that:

* have the same center CS boundary state and CS sampling structure at band ,
* and differ only by neutral moves, inner refinements, or outer embedding changes that do not change band- present content.

Thus:

* **Nested-time matter** provides the **structured content** of a quale (internal and external bands).
* **Gravity** regulates how these structures are arranged and how their ladder profiles (inner, hinge, outer) are shaped (via , horizons, and scale-dependent deviations).
* A **quale** is the **band- projection** of this combined structure, modulo symmetries, at a given context-time.

No phenomenological or psychological claims are needed; qualia are mathematically defined objects built from the same ingredients as the rest of the theory.

**10.5.6 Summary**

Section 10.5 has:

* Defined **present-centered ladder configurations** at a center band , including nested CSs, carriers, and fields across bands.
* Introduced an equivalence relation on these configurations, generated by:
  + neutral moves,
  + inner refinements,
  + outer embedding changes that leave band- observables unchanged.
* Defined a **quale** at band as an **equivalence class** of such configurations under .
* Linked qualia to **CS sampling**: configurations in the same equivalence class yield the same present-plane amplitudes and Born-style probabilities at the center band.
* Clarified how qualia, nested-time matter, and gravity are integrated: qualia are structural summaries of nested ladder configurations centered at the present environment band, not additional entities beyond the ladder.

In the next and final subsection of Part 10 (10.6), we will describe the **coupling between inner (e.g., neural/biological) and outer (gravitational/ambient) ladders**, showing how constraints across bands ensure that nested-time matter and qualia remain consistent with the global gravitational structure.

**10.6 Coupling Between Inner and Outer Ladders**

In practical applications of AR, we often want to distinguish two *interpretive* sides of the same context ladder:

* An **inner ladder** (e.g. “neural/biological”) describing fine-grained, organism-level nested-time matter.
* An **outer ladder** (e.g. “gravitational/ambient”) describing planetary, astrophysical, and cosmic nested-time matter and fields.

Mathematically, they are not two different ladders; they are two *uses* of the **same ladder**, anchored at a common hinge band and coupled by structural constraints.

This section makes that coupling explicit at a structural level.

**10.6.1 Inner and outer ladders as two views of the same structure**

Conceptually, we can factor the full context ladder into:

* An **inner-focused view**: bands and CSs that we interpret as “inside” a given system (e.g. organism, brain, local apparatus).
* An **outer-focused view**: bands and CSs that we interpret as “around” or “containing” that system (environment, planet, galaxy, etc.).

Formally, we consider two overlapping band ranges:

* Inner-oriented bands:
* Outer-oriented bands:

with a shared **center band** (often chosen as the hinge band ).

We can then speak (purely as labels) of:

* an **inner ladder** : the restriction of the ladder to ,
* an **outer ladder** : the restriction to ,

but they are substructures of *one* ladder, and **must match at the shared center band**.

**10.6.2 Matching conditions at the hinge/center band**

Let be the center band for a nested-time system , with:

* Center CS .
* Center band carriers .
* Center band fields (gauge, gravitational, matter) evaluated at .

The **inner ladder** and **outer ladder** must satisfy **matching conditions** at , which can be summarized as:

1. **Boundary state matching**  
   The boundary state of the inner ladder at and that of the outer ladder at coincide:

This ensures that both inner and outer views share *the same* present-moment boundary for the system.

1. **Gauge and matter matching**  
   Gauge and matter fields at band agree when evaluated in inner vs outer descriptions:

up to gauge transformations that act *within* the band.

1. **Gravitational potential matching**  
   The center band gravitational potential and field appearing in the inner and outer ladders must be consistent:

for all relevant boundary points .

These conditions ensure that, at the center band, there is a **single, coherent present environment** that both inner and outer descriptions agree on.

**10.6.3 Source and response consistency**

The **inner ladder** contains detailed nested-time matter (e.g. internal structure, “neural/biological” dynamics); the **outer ladder** encodes larger-scale gravitational and ambient fields. They are coupled structurally via *source–response consistency*:

1. **Inner → outer (source mapping)**  
   Inner nested-time matter patterns induce bandwise source densities . A **source mapping** then produces outer ladder sources:

which feed into the outer gravitational field equations.

1. **Outer → inner (background and constraints)**  
   Outer gravitational potentials and fields impose **constraints** on the admissible inner configurations:

where is the set of inner configurations compatible with the ambient fields (e.g. consistent with horizons, tidal structures, etc.).

1. **Center band self-consistency**  
   At the center band , the source–response mapping must close:
   * The center band source (including contributions from inner structure) feeds the center potential .
   * The resulting must be consistent with both:
     + inner ladder constraints (e.g. stability of the nested-time system),
     + outer ladder asymptotics (e.g. boundary conditions at large scales).

Schematically, the **self-consistency condition** can be written as:

where represents the gravitational response mapping, and is derived from inner nested-time matter.

**10.6.4 Qualia-consistency constraints**

Because **qualia** are defined as equivalence classes of present-centered ladder configurations, the inner/outer coupling must also respect **qualia consistency**:

* For a given quale , every representative must satisfy the inner–outer matching and source–response consistency at the center band.

More concretely:

1. **Inner–outer compatibility of present content**  
   The present content at band (CS sampling structure, gravitational potential, and local gauge/matter configuration) must be recoverable *either* from:
   * the inner ladder plus its couplings,
   * or the outer ladder plus its couplings,  
     and both reconstructions must yield the **same** band- data within that equivalence class.
2. **Neutral-move robustness**  
   Neutral moves (tick-level and ladder-level) that change inner or outer details but preserve band- observables do *not* change the quale. Thus, qualia are invariant under reconfigurations of inner or outer bands that remain consistent with the inner–outer matching conditions.
3. **No conflicting global structure**  
   It is not allowed (in a physically consistent AR model) to have a configuration where:
   * inner structure demands one gravitational configuration at the center band,
   * while outer ladder gravitational structure imposes an incompatible one,  
     because such a configuration would fail the source–response consistency and thus fail to be a member of any *single* quale class (it would violate the basic structural axioms).

Qualia are therefore defined only on **globally consistent** inner–outer coupled configurations.

**10.6.5 Structural locality and limited couplings**

Inner and outer ladders are not free to couple arbitrarily at all bands; the AR structure enforces a kind of **structural locality**:

1. **Gate-weighted coupling strength**  
   The **pivot weights** modulate how strongly bands couple:
   * Bands near the hinge (where ) typically have and thus strong coupling.
   * Deep inner and far outer bands (with far from 2) are effectively damped in their influence on the center band.
2. **Finite coupling range in band space**  
   Collapse/expansion operators primarily connect **neighboring bands**. Longer jumps along the ladder are built from compositions of these steps and are therefore weaker or more constrained. This implies:
   * Inner details at very negative influence the center only via intermediate bands.
   * Outer structures at very large influence the center through a chain of intermediate couplings.
3. **Emergent local laws**  
   Because of this band-local structure, **effective local laws** emerge at the center band:
   * Gravity at the hinge behaves like a local field theory in the 4D thickened picture, even though it is ultimately sourced by a structured ladder.
   * Inner nested-time dynamics (e.g. “neural” processes) appear to be coupled locally to fields at the center band, with outer ambient influences entering mainly through slowly varying background fields and boundary conditions.

This structural locality ensures that:

* The inner ladder is not “instantaneously” sensitive to distant outer structures in arbitrary ways,
* Qualia at the center band depend primarily on **near-hinge** inner and outer bands, in line with the idea of a localized present environment.

**10.6.6 Summary**

Section 10.6 has:

* Distinguished **inner** and **outer** views of the context ladder as two restricted perspectives on the same structure, anchored at a common center band (often the hinge).
* Introduced **matching conditions** at the center band for boundary states, gauge/matter fields, and gravitational potentials, ensuring a single coherent present environment.
* Described **source–response consistency** between inner nested-time matter and outer gravitational/ambient fields, with the center band as the closure point of the mapping.
* Shown that **qualia** are equivalence classes only over **consistent** inner–outer coupled configurations: any conflict between inner-sourced and outer-imposed present content would violate the structural axioms and thus not define a valid quale.
* Emphasized **structural locality** in band space: pivot weights and nearest-neighbor ladder couplings limit how strongly deep inner and far outer bands influence the center, underpinning the emergence of effective local laws at the hinge.

With this, Part 10 (“Gravitational Sector & Nested-Time Matter”) is structurally complete. The AR V1 theory now has:

* a scalar gravitational sector anchored at an area-law hinge,
* scale-dependent deviations, horizon and entropy structure,
* a notion of nested-time matter and worldlines,
* a structural definition of qualia,
* and explicit coupling constraints between inner and outer ladder views—all without invoking an external spacetime or separate matter substrate.

**11. Collapse Kernels, Light-Cone Geometry & Unified Inverse-Square Law**

**11.1 Preliminaries: (D(n)), (g(D)), Collapse Operators**

Before we introduce explicit collapse kernels on the hinge boundary and extend them to null-cone geometry, we collect the key ingredients already developed and sharpen a few assumptions. This section fixes:

* the **bandwise dimension profile** (D(n)) and pivot weights (g\_n = g(D(n))),
* the **boundary state spaces** (\mathcal{H}\_n^\partial),
* and a class of **collapse operators** (K\_n) that act on those spaces and on the ladder.

These will be used in Sections 11.2–11.7 to define:

* a collapse kernel on an S²-like hinge boundary,
* its generalization to a 4D null hypersphere,
* and the unified inverse-square law at the hinge.

**11.1.1 Bandwise dimension profile (D(n)) and pivot weights (g\_n)**

From Part VI, recall the **discrete dimension curve** (D : \mathbb{Z}\to [D\_{\min},D\_{\max}]) with:

* Hinge condition:  
  [  
  D(0) = 2,  
  ]
* Inner asymptotics:  
  [  
  \lim\_{n\to -\infty} D(n) = D\_{\text{in}} \in (2,3],  
  ]
* Outer asymptotics:  
  [  
  \lim\_{n\to +\infty} D(n) = D\_{\text{out}} \in [1,2),  
  ]
* Monotonicity in the large: (D(n)) decreases as (n) increases, interpolating from (D\_{\text{in}}) to (D\_{\text{out}}).

The **pivot function** (g(D)) is a positive, continuous function on ([D\_{\min},D\_{\max}]) with:

* Normalization at the hinge:  
  [  
  g(2) = 1,  
  ]
* Local regularity near (D=2),
* Boundedness: (g\_{\min} \le g(D)\le g\_{\max}).

We define the **bandwise pivot weights** as  
[  
g\_n := g(D(n)),\quad n\in\mathbb{Z}.  
]

These weights appear throughout:

* in the master action,
* in gauge and gravitational sectors,
* and now, in weighted sums of bandwise collapse kernels and null-cone operators.

The band (n=0) is the **pivot band** where  
[  
D(0)=2,\quad g\_0 = 1,  
]  
and everything is normalized.

**11.1.2 Boundary state spaces (\mathcal{H}\_n^\partial)**

For each band (n), the boundary graph (\mathcal{G}\_n = (V\_n,E\_n)) has an associated **boundary state space**  
[  
\mathcal{H}\_n^\partial,  
]  
which we can think of as:

* complex-valued functions on (V\_n),
* or more general structured states (e.g. spinor-valued, representation-valued) as needed.

Here we only require:

1. **Vector space structure**  
   (\mathcal{H}\_n^\partial) is a complex vector space.
2. **Inner product**  
   There exists a positive-definite inner product  
   [  
   \langle \cdot,\cdot\rangle\_n : \mathcal{H}\_n^\partial \times \mathcal{H}*n^\partial \to \mathbb{C},  
   ]  
   typically induced by node weights (\mu\_n(v)>0):  
   [  
   \langle f,g\rangle\_n  
   = \sum*{v\in V\_n} \mu\_n(v), f(v)^\* g(v).  
   ]
3. **Complete orthonormal basis**  
   Each (\mathcal{H}*n^\partial) admits an orthonormal basis ({e*{n,\alpha}}\_\alpha) with respect to (\langle\cdot,\cdot\rangle\_n).

In the continuum limit (for sufficiently fine (\mathcal{G}\_n)), (\mathcal{H}\_n^\partial) can be regarded as a function space on a surface approximating the boundary of a region at band (n). For now, the discrete picture is sufficient.

**11.1.3 Bandwise collapse operators (K\_n)**

We now introduce **collapse operators** that act on the boundary state spaces (\mathcal{H}\_n^\partial). These are not the same as the collapse/expansion maps between bands (Section 6.2); rather, they are **within-band projective operators** that:

* damp or average over fine boundary structure,
* and, at the hinge, reduce to a particularly symmetric form.

**Definition 11.1.1 (Bandwise collapse operator).**  
For each band (n\in\mathbb{Z}), a **collapse operator** is a linear map  
[  
K\_n : \mathcal{H}\_n^\partial \to \mathcal{H}\_n^\partial  
]  
satisfying:

1. **Self-adjointness**  
   [  
   \langle f, K\_n g\rangle\_n = \langle K\_n f, g\rangle\_n  
   \quad\text{for all } f,g\in\mathcal{H}\_n^\partial.  
   ]
2. **Positivity**  
   [  
   \langle f, K\_n f\rangle\_n \ge 0 \quad\text{for all } f\in\mathcal{H}\_n^\partial.  
   ]
3. **Boundedness**  
   (K\_n) is bounded as an operator on (\mathcal{H}\_n^\partial).
4. **Normalization at the hinge**  
   At the pivot band (n=0), (K\_0) is a nontrivial projector whose structure will be specified in 11.2; key properties include:
   * Idempotence:  
     [  
     K\_0^2 = K\_0,  
     ]
   * Rank-1 (or finite-rank) character in the simplest realization, projecting onto a “constant” or “spherically symmetric” boundary mode.

For (n\neq 0), (K\_n) may not be a strict projector; it may be:

* a smoothing operator (e.g. a spectral filter on (\mathcal{G}\_n)),
* or any positive, self-adjoint operator that encodes band-(n) collapse behavior.

In all cases, the spectrum of (K\_n) is assumed to lie in ([0,1]), i.e., it acts as a contraction in norm.

**11.1.4 Spectral decomposition of (K\_n) and hinge simplification**

Since each (K\_n) is self-adjoint and positive, it has a spectral decomposition  
[  
K\_n = \sum\_\alpha \lambda\_{n,\alpha}, |u\_{n,\alpha}\rangle\langle u\_{n,\alpha}|,  
]  
where:

* (|u\_{n,\alpha}\rangle) are orthonormal eigenvectors in (\mathcal{H}\_n^\partial),
* (\lambda\_{n,\alpha}\in[0,1]) are eigenvalues.

We assume the following **hinge simplification**:

**Axiom 11.1.2 (Hinge spectral form).**  
At band (n=0), the collapse operator (K\_0) has:

1. A unique **maximal eigenvalue** (\lambda\_{0,0} = 1),
2. All other eigenvalues strictly less than 1:  
   [  
   \lambda\_{0,\alpha} < 1 \quad\text{for }\alpha\neq 0,  
   ]
3. The corresponding eigenvector (|u\_{0,0}\rangle) is **constant on the boundary** in an appropriate sense (e.g. (u\_{0,0}(v) = \text{const}) for all (v\in V\_0)).

Thus:  
[  
K\_0 = |u\_{0,0}\rangle\langle u\_{0,0}|  
+ \sum\_{\alpha\neq 0} \lambda\_{0,\alpha},|u\_{0,\alpha}\rangle\langle u\_{0,\alpha}|.  
]

In the idealized **pivot limit**, we may treat all (\lambda\_{0,\alpha\neq 0}=0), so that  
[  
K\_0 \approx |u\_{0,0}\rangle\langle u\_{0,0}|,  
]  
a rank-1 projector onto the constant mode. This is the simplified hinge identity we will exploit later.

Away from the hinge, the spectral structure of (K\_n) is more general, but we assume (\lambda\_{n,\alpha}\to 0) for “high frequency” modes, so that (K\_n) acts as a smoother.

**11.1.5 Ladder-averaged collapse operators**

We often want to combine the bandwise (K\_n) into ladder-level operators, either to:

* act on a direct sum (\bigoplus\_n \mathcal{H}\_n^\partial), or
* define effective kernels after summing over bands with weights (g\_n).

Let  
[  
\mathcal{H}*\partial^{\text{ladder}}  
:= \bigoplus*{n\in\mathbb{Z}} \mathcal{H}\_n^\partial.  
]

We define a **ladder collapse operator** (K\_{\text{ladder}}) by

[  
K\_{\text{ladder}} := \bigoplus\_{n} K\_n,  
]  
so that for a ladder boundary state (\Psi = {\psi\_n}*n),  
[  
(K*{\text{ladder}} \Psi)\_n = K\_n \psi\_n.  
]

We can also define a **pivot-weighted ladder collapse** by inserting the pivot weights:

[  
K\_{\text{pivot}} := \sum\_{n} g\_n, \pi\_n K\_n,  
]  
where (\pi\_n) is the projector onto the (n)-th band component in (\mathcal{H}\_\partial^{\text{ladder}}). Concretely,

* For (\Psi = {\psi\_n}*n),  
  [  
  (K*{\text{pivot}} \Psi)\_n = g\_n,K\_n\psi\_n.  
  ]

This weighted operator will be useful when we construct:

* ladder-level null-cone kernels,
* and unified inverse-square fields, which must incorporate both the bandwise collapse behavior and the pivot weighting.

**11.1.6 Compatibility with the boundary projector (\mathcal{B})**

Recall the **boundary projector** (\mathcal{B}) from Section 5.3, which collapses radial structure to the hinge layer. At the level of boundary state spaces, (\mathcal{B}) acts like a map  
[  
\mathcal{B}*\partial : \mathcal{H}*\partial^{\text{ladder}} \to \mathcal{H}*0^\partial,  
]  
with  
[  
\mathcal{B}*\partial({\psi\_n}\_n) = \psi\_0^{\text{eff}},  
]  
where (\psi\_0^{\text{eff}}) is the **hinge-effective boundary state**, obtained by collapsing radial contributions.

We impose a **compatibility condition** between (K\_{\text{ladder}}) and (\mathcal{B}\_\partial):

**Axiom 11.1.3 (Hinge projection–collapse compatibility).**  
For ladder boundary states in a suitable class, we require:

1. Hinge-first then collapse:  
   [  
   K\_0,\mathcal{B}*\partial(\Psi)  
   ;\approx; \mathcal{B}*\partial\bigl(K\_{\text{ladder}} \Psi\bigr).  
   ]
2. In particular, for states whose band support is concentrated near the hinge and whose high-frequency modes at (n=0) are heavily damped by (K\_0), we may approximate  
   [  
   K\_0,\mathcal{B}*\partial(\Psi)  
   \approx \mathcal{B}*\partial(\Psi),  
   ]  
   i.e. (K\_0) acts as **effective identity** on the hinge-projected state (since it projects onto the constant mode and rapidly suppresses deviations).

In the pivot limit (rank-1 (K\_0)), the hinge-projected state is effectively replaced by the constant mode amplitude; this is the sense in which the hinge collapse kernel becomes an “identity” on the pivot shell in later sections.

**11.1.7 Summary**

Section 11.1 has:

* Recalled the **bandwise dimension profile** (D(n)) and pivot weights (g\_n = g(D(n))), with pivot band (n=0) where (D(0)=2), (g\_0=1).
* Fixed **boundary state spaces** (\mathcal{H}\_n^\partial) and their inner products.
* Introduced **bandwise collapse operators** (K\_n), self-adjoint and positive on (\mathcal{H}\_n^\partial), with a special **pivot structure** at (n=0) where (K\_0) has a unique maximal eigenvalue and a constant-mode eigenvector.
* Defined **ladder-level** and **pivot-weighted** collapse operators that act on the direct sum of boundary state spaces.
* Stated a **compatibility condition** between the ladder collapse operators and the boundary projector (\mathcal{B}\_\partial), ensuring that hinge projection and collapse interplay in a controlled, pivot-centric way.

These preliminaries set the stage for:

* defining an explicit **S²-like collapse kernel** on the hinge boundary in Section 11.2,
* generalizing it to a **4D null-cone kernel** and **retarded composite moment operator** in later subsections,
* and deriving a **unified inverse-square law** for appropriate scalar fields at the pivot.

**11.2 S² Collapse Kernel & 0-Context Pivot Identity**

In this section we specialize the hinge boundary (band (n=0)) to an **S²-like surface**, define a **collapse kernel** on it, and show how, at the pivot (D(0)=2), this kernel reduces to a **constant-mode projector**. This is the sense in which the **0-context** (hinge band) is a unique pivot: after collapse, angular structure becomes invisible and only a uniform scalar “present shell” remains.

We work in a continuum-flavored language for clarity (S² and spherical harmonics), with the understanding that the discrete boundary graph (\mathcal{G}\_0) is an approximation to such a surface.

**11.2.1 Hinge Boundary as an S²-Like Surface**

At the hinge band (n=0), the boundary graph (\mathcal{G}\_0=(V\_0,E\_0)) has effective dimension (D(0)=2), with area-law scaling. In the continuum approximation, we model this as a 2-sphere:

* Denote the hinge boundary as a surface (\mathcal{S}\_0 \simeq S^2).
* Points on (\mathcal{S}\_0) are labeled by angular coordinates (x=(\theta,\varphi)).
* The natural measure is the area element (\mathrm{d}\Omega(x)) on (S^2).

The discrete boundary state space (\mathcal{H}\_0^\partial) can be seen as an approximation to a function space on (\mathcal{S}\_0), e.g. square-integrable functions (f:\mathcal{S}\_0\to\mathbb{C}) with inner product

[  
\langle f,g\rangle\_0 = \int\_{\mathcal{S}\_0} f(x)^\* g(x),\mathrm{d}\Omega(x).  
]

In this continuum picture, a **collapse operator** (K\_0) is naturally written as an integral operator with kernel (K\_0(x,y)),

[  
(K\_0 f)(x) = \int\_{\mathcal{S}\_0} K\_0(x,y),f(y),\mathrm{d}\Omega(y).  
]

Our goal is to specify structural properties of (K\_0(x,y)) and, in particular, its behavior at the pivot dimension (D=2).

**11.2.2 Rotationally Invariant Collapse Kernel**

We require the hinge collapse kernel to be **rotationally invariant** on (\mathcal{S}\_0). This means:

* (K\_0(x,y)) depends only on the **geodesic distance** between (x) and (y):  
  [  
  K\_0(x,y) = \mathcal{K}(\cos\gamma(x,y);\delta\_0),  
  ]  
  where (\gamma(x,y)) is the angle between (x) and (y) on (S^2), and (\delta\_0) is a parameter encoding how far we are from the ideal pivot limit.
* Equivalently, writing (\mu = \cos\gamma \in [-1,1]),  
  [  
  K\_0(x,y) = \mathcal{K}(\mu;\delta\_0).  
  ]

Rotational invariance implies that (K\_0) commutes with the action of the rotation group SO(3) on functions on (\mathcal{S}\_0). That is,

[  
K\_0 \circ R = R \circ K\_0  
]  
for any rotation (R) acting on (f(x)) as ((Rf)(x) = f(R^{-1}x)).

This symmetry heavily constrains the spectral form of (K\_0).

**11.2.3 Spectral Representation on Spherical Harmonics**

The spherical harmonics ({Y\_{\ell m}(x)}) form an orthonormal basis of (L^2(S^2)) and diagonalize any rotationally invariant kernel. Specifically,

* For (\ell=0,1,2,\dots), (m=-\ell,\dots,\ell),
* The spherical harmonics satisfy  
  [  
  \int\_{S^2} Y\_{\ell m}(x)^\*,Y\_{\ell' m'}(x),\mathrm{d}\Omega(x)  
  = \delta\_{\ell\ell'}\delta\_{mm'}.  
  ]

For a rotationally invariant kernel (K\_0(x,y)), there exists a sequence of **mode weights** ({\lambda\_\ell}\_{\ell=0}^\infty) such that

[  
(K\_0 Y\_{\ell m})(x)  
= \lambda\_\ell,Y\_{\ell m}(x),  
]  
independent of (m).

Thus, we may write

[  
K\_0(x,y)  
= \sum\_{\ell=0}^\infty \lambda\_\ell,\sum\_{m=-\ell}^{\ell} Y\_{\ell m}(x) Y\_{\ell m}(y)^\*.  
]

Using the addition theorem for spherical harmonics,

[  
\sum\_{m=-\ell}^{\ell} Y\_{\ell m}(x)Y\_{\ell m}(y)^\*  
= \frac{2\ell+1}{4\pi}P\_\ell(\cos\gamma(x,y)),  
]  
where (P\_\ell) are the Legendre polynomials, this becomes

[  
K\_0(x,y)  
= \sum\_{\ell=0}^\infty \lambda\_\ell,\frac{2\ell+1}{4\pi}P\_\ell(\cos\gamma(x,y)).  
]

The operator is:

* **Self-adjoint** if all (\lambda\_\ell\in\mathbb{R}),
* **Positive** if all (\lambda\_\ell\ge 0),
* **Contractive** if (0\le\lambda\_\ell\le 1) for all (\ell).

This matches the generic spectral form from Section 11.1, with the eigenvectors (|u\_{0,\alpha}\rangle) identified with the spherical harmonics and (\lambda\_{0,\alpha}=\lambda\_\ell).

**11.2.4 Pivot Condition at (D=2): Constant-Mode Projector**

At the hinge, we want the collapse kernel to encode the **2D pivot** in a sharp way. Structurally, this means:

* At the **pivot dimension** (D=2), collapse eliminates all **non-constant modes**, leaving only the uniform (monopole) component.
* The constant mode should be preserved with eigenvalue 1.

The constant spherical harmonic is

[  
Y\_{00}(x) = \frac{1}{\sqrt{4\pi}},  
]  
and it is the unique rotationally invariant mode.

We therefore impose the **pivot eigenvalue pattern** at (D=2):

1. (\lambda\_0 = 1) for the monopole mode ((\ell=0)),
2. (\lambda\_\ell = 0) for all (\ell\ge 1) in the **ideal pivot limit**.

Then

[  
K\_0 f  
= \sum\_{\ell m} \lambda\_\ell \langle Y\_{\ell m}, f\rangle\_0,Y\_{\ell m}  
= \langle Y\_{00}, f\rangle\_0,Y\_{00}  
= \left( \frac{1}{4\pi}\int\_{S^2} f(y),\mathrm{d}\Omega(y) \right)\cdot 1,  
]  
i.e. (K\_0) sends any function (f) to its **uniform average** over the sphere.

In this limit, (K\_0) is a rank-1 projector:

[  
K\_0^2 = K\_0,\qquad  
\mathrm{Im}(K\_0) = \mathrm{span}{Y\_{00}}.  
]

This realizes precisely the hinge spectral structure anticipated in Section 11.1:

* unique maximal eigenvalue 1,
* all other eigenvalues strictly less (here: 0),
* corresponding eigenvector equal (up to normalization) to a constant function on the boundary.

We call this the **0-context pivot identity**:

**0-context pivot identity:**  
At the hinge band, with (D(0)=2) and in the ideal pivot limit, the collapse kernel (K\_0) is the projector onto the uniform boundary mode. All non-uniform angular structure is erased by collapse, so the hinge sees only a **single scalar amplitude** on its boundary, independent of angular location.

**11.2.5 Deviation from the Pivot: (\lambda\_\ell(D))**

Away from the pivot dimension, the spectrum ({\lambda\_\ell}) is deformed by the effective dimension (D) (or equivalently, by some deviation parameter (\delta) with (D=2+\delta)). Structurally, we require:

* (\lambda\_0(D) = 1) for all admissible (D): the constant mode is preserved, reflecting the basic consistency of the hinge average.
* For (\ell\ge 1), (\lambda\_\ell(D)\in[0,1)), with:
  + (\lambda\_\ell(D)\to 0) as (D\to 2), for each fixed (\ell\ge 1),
  + monotone behavior in (|D-2|): the farther from the pivot, the less the kernel suppresses higher-(\ell) modes, reflecting the fact that **off-pivot bands preserve more angular detail**.

We can summarize this as:

[  
\lambda\_\ell(D)

\begin{cases}  
1, & \ell = 0,\[4pt]  
f\_\ell(D), & \ell \ge 1,  
\end{cases}  
]  
where (f\_\ell(D)) is continuous in (D), satisfies (f\_\ell(2)=0), and (0\le f\_\ell(D)<1) otherwise.

In the discrete ladder, this dependence on (D(n)) means that **bandwise collapse operators** (K\_n) interpolate between:

* a pure uniform projector at the hinge ((n=0)),
* less suppressive kernels at bands with (D(n)\neq 2), where more angular structure can survive.

The key point is that the **pivot band** (n=0) is structurally singled out by the collapse kernel: only there does the kernel reduce to a constant-mode projector.

**11.2.6 Relation to Bandwise Collapse (K\_n) and the Ladder**

The S² collapse kernel discussed here is the **continuum prototype** of the bandwise collapse operators (K\_n) on the discrete ladder:

* At band (n=0), (K\_0) is modeled by the rotationally invariant rank-1 projector onto the constant mode. Discretely, this corresponds to assigning the same value to every node in (V\_0) equal to the average of the input state.
* At bands (n\neq 0), (K\_n) are modeled as rotationally invariant but less stringent smoothers, retaining higher-(\ell) components ((\lambda\_\ell(D(n))\in (0,1)) for some (\ell\ge 1)).

When combined with:

* the boundary projector (\mathcal{B}\_\partial) (which collapses radial structure onto band (0)), and
* the ladder-collapse operators (K\_{\text{ladder}}) and (K\_{\text{pivot}}),

we get the following structural behavior:

1. **Hinge projection then collapse**  
   For states whose support is primarily near the hinge and not extremely high-frequency in angle,  
   [  
   K\_0,\mathcal{B}*\partial(\Psi)  
   \approx \mathcal{B}*\partial(\Psi),  
   ]  
   i.e. hinge projection is effectively already in the constant mode on the boundary.
2. **Double-flip + projection invariance**  
   As anticipated in Section 5.4, certain balanced inward/outward flip sequences, followed by hinge projection and collapse, yield the same uniform boundary state. This is a direct consequence of the fact that (K\_0) projects to the constant mode: any symmetric angular perturbation is “averaged out” at the pivot.
3. **Uniqueness of 0-context pivot**  
   No other band admits such a collapse kernel that reduces exactly to a constant-mode projector; at bands with (D(n)\neq 2), more angular modes survive collapse. Hence, the **0-context** is uniquely singled out as the **isotropic pivot** in the ladder.

**11.2.7 Summary**

Section 11.2 has:

* Modeled the hinge boundary at band (n=0) as an S²-like surface (\mathcal{S}\_0) with rotational symmetry.
* Defined a rotationally invariant collapse kernel (K\_0(x,y)) and expressed it in a spectral basis of spherical harmonics.
* Imposed a **pivot condition** at (D(0)=2): (K\_0) becomes a **constant-mode projector**, preserving only the uniform mode and erasing all non-uniform angular structure.
* Interpreted this as the **0-context pivot identity**: after collapse at the hinge, the boundary reduces to a single scalar degree of freedom, independent of angular position.
* Related this S² kernel to bandwise discrete collapse operators (K\_n) and to the ladder-level collapse structure, emphasizing the uniqueness of the hinge band in this regard.

In the next subsection (11.3), we will extend this hinge S² collapse structure to a **null-cone shell** in 4D, introducing a **4D hyperspherical collapse kernel** and a **retarded composite moment operator** that becomes identity at the pivot, thereby connecting the spatial hinge picture to light-cone geometry.

**11.3 Mirror Symmetry & Pivot-Location Functionals**

In Sections 5–6 we introduced the **dimension profile** on the ladder and, for convenience, often chose a **logistic-type** ansatz with a hinge at . In this section we make that hinge characterization more explicit by:

* defining **mirror-symmetry functionals** on that measure how “centered” a candidate pivot index is, and
* showing that, for an ideal logistic profile with a true pivot at , these functionals are uniquely minimized at .

This gives a purely structural way to **locate the pivot band** from a given (or any symmetric approximation to it), independent of any collapse or field data.

**11.3.1 Ideal mirror-symmetric dimension curve**

We recall the idealized **logistic** profile from 6.5, written here as to emphasize that it is a “target” or “model” curve:

with the hinge chosen at and parameters constrained so that

Under this last condition one has the exact **mirror relation**

Equivalently,

Thus is a **mirror-symmetry center** for : deviations from 2 at are equal in magnitude and opposite in sign.

In practice, a measured or simulated dimension profile will deviate from . We now define functionals that quantify how symmetric is around a **candidate pivot** , and then examine their behavior for the ideal case.

**11.3.2 Raw mirror-symmetry error**

Let be any bandwise dimension profile, and fix a **finite window** of radius . For a candidate pivot index , we define a **raw mirror error functional**:

where are weights (for example or decaying with ).

Interpretation:

* For each offset , the expression

measures **how far** the profile deviates from being symmetric with respect to a mirror centered at .

* Squaring and summing (with weights) yields a **non-negative scalar** .
* If were **exactly symmetric** around in the sense

then .

Thus is a raw measure of how “mirror-centered” the chosen pivot is, within the band window .

**11.3.3 Corrected error (optional trend removal)**

In more general contexts there may be a **slow drift** or **trend** in that we might want to factor out before assessing symmetry. One simple way to do this is to subtract a local linear fit around .

Let be the least-squares linear approximation to over . Define the **detrended profile**:

We then define the **corrected mirror error** as

Notice we no longer subtract explicitly: by construction, the linear fit has been removed and any remaining symmetry should present as .

In the ideal mirror-symmetric case, where is exactly symmetric around plus at most a linear trend, we get .

For the purposes of this V1 volume, it is enough to state the raw version ; the corrected version simply refines robustness to slow drifts. We will focus on below.

**11.3.4 Unique minimum at the hinge for the ideal logistic profile**

We now show that, for the ideal logistic profile with mirror symmetry around 0,

* the raw error is **exactly zero** at for any integer window radius , and
* it is **strictly positive** for all (for any nontrivial window), assuming a strictly monotone logistic curve.

**Proposition 11.3.1 (Hinge as unique minimizer of in the ideal case).**

Let be an ideal logistic profile with:

1. ,
2. for all ,
3. strictly decreasing in .

Then for any finite window radius and weights :

1. .
2. For any integer such that , we have .

*Sketch of proof.*

1. **At .**  
   For every , mirror symmetry gives

Since , the bracket becomes

Thus each term in the sum is zero, and .

1. **At .**  
   Suppose for a contradiction that for some . Then

This says that the **mirror image** of around is symmetric in the sense used above.

But is strictly decreasing in . For a strictly monotone logistic curve, the only integer at which such a perfect mirror symmetry can hold over more than the trivial window is the true symmetry center; in our assumptions this is .

More concretely, for , consider . Then the symmetry condition gives

Using strict monotonicity, one can check that

which is incompatible with the stepwise mirror relation for a function that is exactly mirror-symmetric only about 0. A similar argument applies for .

Hence no can satisfy all the mirror equalities for a nontrivial window, and at least one term in the sum defining must be strictly positive, so .

Thus, in the ideal logistic case, **the only exact symmetry center is** , and the raw mirror error functional has a unique minimum at this hinge band.

**11.3.5 Use as an abstract pivot-locator**

In realistic situations, a dimension profile may be:

* noisy,
* only approximately logistic,
* defined only on a finite band range.

The mirror error functionals and then become **abstract tools**:

* One scans candidate pivots and computes for a fixed window .
* The **pivot band** is identified as the integer that **minimizes** (or if drift is present).

In the ideal symmetric logistic model, this minimum is **exactly at** the true hinge . For small deviations from the ideal, the minimizer remains close to 0 under stability assumptions (small perturbations of a strictly convex error function).

Put differently:

The mirror error functionals provide a **band-intrinsic pivot locator**, relying only on the dimension curve . No reference to collapse operators, fields, or external calibration is needed to identify the hinge as the unique mirror center.

This is conceptually consistent with — and complementary to — the hinge characterization via:

* **collapse kernels** (11.2, 11.3),
* **area-law behavior** (6.6, 10.1),
* and **UGM scaling** (8.2, 11.5).

All of these are different faces of the same hinge structure; the mirror-symmetry functionals are simply the **purely geometric, bandwise** way to see that is the pivot.

**11.4 Null-Cone Shell as Fractal Surface & 4D Hyperspherical Collapse Kernel**

We now extend the hinge S² collapse structure of Section 11.2 into a **null-cone** setting. The idea is:

* Use the **invariant interval** to define null separations (zero proper-time).
* Treat the **null-cone shell** as a fractal surface of effective dimension 2, closely related to the hinge S².
* Define a **4D hyperspherical collapse kernel** on this null shell.
* Build from it a **retarded composite moment operator** that becomes identity (or a constant-mode projector) at the pivot, mirroring the S² hinge behavior in a light-cone geometry.

This sets up the tools needed to unify inverse-square behavior and null propagation in later subsections.

**11.4.1 Null separation from the invariant interval**

Recall the invariant interval derived from flip counts (Section 3.4):

with:

* a coordinate-time increment,
* a proper-time increment,
* a spatial separation magnitude,
* the unit conversion factor .

A **null separation** is defined by the condition

In emergent 4D terms, we view:

* events (or context configurations) related by such a null separation as lying on a **light-cone** relative to each other:
  + “outgoing” null: ,
  + “ingoing” null: .

The set of configurations null-related to a given hinge event thus forms a **3D null hypersurface** (the usual light-cone), whose spatial cross-sections at fixed are 2-spheres of radius .

This is the continuum picture we now connect to the hinge S² collapse.

**11.4.2 Null-cone shell as effective fractal surface**

At a fixed coordinate-time separation , the spatial cross-section of the outgoing null cone is a 2-sphere of radius . Structurally:

* This cross-section inherits the **S² geometry** of the hinge boundary, scaled by .
* As a set of boundary directions (angular data only), it has effective dimension .

We therefore treat the **null-cone shell** at fixed as:

* a **fractal surface of effective dimension 2**,
* with angular structure described by the same spherical-harmonic decomposition used at the hinge,
* and radial dependence encoded by the scale .

In other words:

The null cone, at fixed radius , is an S²-like shell whose angular collapse properties are governed by the same class of kernels as the hinge S², while radial positioning is controlled by the invariant relation and the context-time structure.

This motivates defining a **factorized kernel** on the null shell: one part for radial/time structure, one for angular structure.

**11.4.3 4D null-shell kernel: radial δ and angular S² kernel**

We now define a **4D hyperspherical collapse kernel** acting on functions defined on the null cone. Let:

* be a null-cone “radius” variable (with dimension of length in internal units).
* be an angular coordinate on S² (the direction of propagation).
* Points on the null cone are labeled by with for the outgoing cone.

Let be a scalar field defined on the outgoing null cone. We define a **null-shell kernel** as

For a **retarded, radius-preserving** collapse, a structurally natural form is:

where:

* enforces that collapse acts on each radial shell independently (no mixing between different null radii in this simplest model),
* is the S² collapse kernel from Section 11.2, acting on the angular variables.

Thus,

This has several important features:

1. **Angular-only collapse**:  
   At each fixed null radius , the kernel collapses angular structure according to , leaving radial dependence untouched.
2. **S² spectral structure at each shell**:  
   The spherical harmonic expansion applies at each , with eigenvalues as in Section 11.2.
3. **Pivot behavior at** :  
   At the hinge/pivot, reduces to a constant-mode projector, so sends any angular dependence at fixed to its uniform average on the sphere.

This is the structural reason we can treat the null-cone shell as having an effective **dimension 2** in the angular directions, with collapse erasing non-uniform angular structure at the pivot.

**11.4.4 Retarded composite moment operator**

To connect collapse on the null cone to **sources** inside the cone, we define a **retarded composite moment operator** that maps volume (or ladder) sources into collapsed null-shell data.

Consider a scalar source density in the emergent 4D domain (or a more general ladder source that reduces to such a scalar under thickening). The null cone through a point at time is determined by all points with

i.e. points on (or inside, for a retarded kernel) the past light cone. We write this schematically as .

The **retarded composite moment operator** maps to a null-shell density on , and then collapses angular structure via . Structurally:

where:

* is the past region of interest (e.g. past light cone of the observation point),
* is a retarded kernel enforcing causal ordering and null-cone geometry (e.g. a Green’s function-like distribution that concentrates support near the null shell).

The **composite moment operator** is then

At the pivot, with the constant-mode projector form of , this reduces to an **isotropic average** of the retarded null-shell density:

i.e. it becomes independent of . The null-shell data seen at the pivot is then:

* a single scalar function of (radius or null time),
* encoding the **total retarded “moment”** of the source distributed over the angular directions.

This is the **null-cone analogue** of the hinge S² constant-mode projector: in both cases, after collapse, only a uniform scalar survives.

**11.4.5 Pivot identity in null-cone context**

We can now state the **pivot identity** for the null-cone collapse:

**Null-cone pivot identity:**  
At the hinge dimension , with S² collapse kernel as a constant-mode projector, the 4D hyperspherical collapse kernel and the retarded composite moment operator reduce (at each null radius ) to isotropic averages of incoming null-shell data. All directional dependence is erased; only scalar, radius-dependent moments of the source survive.

Formally, at the pivot:

* For any ,

independent of .

* therefore acts as an operator mapping to a scalar function , which can be further related to radial potentials and fields (e.g. inverse-square behavior) as developed later.

This identity is the null-cone counterpart of the S² **0-context pivot identity** and is crucial for:

* tying null propagation (light-cone geometry) to the same 2D pivot that underlies the inverse-square law,
* and showing that, at the pivot, scalar fields sourced by **retarded** data on the null cone inherit an **isotropic inverse-square structure** in the radial direction.

**11.4.6 Summary**

Section 11.4 has:

* Defined null separations using the invariant interval and identified the **null cone** as a 3D hypersurface in emergent 4D.
* Treated the null-cone shell at fixed radius as an S²-like fractal surface of effective dimension 2, sharing the same angular structure as the hinge boundary.
* Defined a **4D hyperspherical collapse kernel** that factorizes into a radial δ and an angular S² kernel , acting on functions defined on the null cone.
* Introduced a **retarded composite moment operator** that maps sources to collapsed null-shell data; at the pivot, this operator reduces to an isotropic average over the sphere at each null radius.
* Stated the **null-cone pivot identity**: at , null-cone collapse erases angular structure and retains only scalar, radius-dependent source moments.

In the next subsection (11.4), we will build on this null-cone pivot structure to define **relativistic context flips**, interpret time dilation and context “rapidity” in terms of flip sequences, and make explicit how the pivot-based collapse structure supports an emergent special-relativistic kinematics.

**11.5 Relativistic Context Flips & Twin Paradox**

With the null-cone pivot structure in place, we now make explicit how **special-relativistic kinematics** arises from:

* the **flip algebra** (Sections 2–3),
* the **context ladder** and CS frames, and
* the **null-cone / hinge pivot identities** (Sections 11.2–11.3).

The key idea is:

Worldlines are **equivalence classes of flip sequences** between CS frames; “velocity” and “rapidity” are **context parameters** describing how those flip patterns distribute between temporal and spatial components, and time dilation is the ratio between pivot factors for different frames.

We close with a ladder-level representation of the **twin paradox** and relate it back to the collapse structure of the null cone.

**11.5.1 Worldlines as flip sequences between CS frames**

Recall:

* A **CS frame** is a Collective Sphere (CS) with shared boundary configuration and synchronized tick structure (Section 3.5).
* A **flip word** is a sequence of primitive operators that maps one carrier to another, with a flip-count vector .
* The **invariant interval** associates with , , , obeying

We interpret a **worldline** between CS frames as an equivalence class of flip sequences between two chosen CSs.

**Definition 11.4.1 (Context worldline).**  
Fix two CS frames, and . A **context worldline** from to is the equivalence class

where is any admissible flip word that maps a carrier in to a carrier in , and is flip-count equivalence (Section 2.4).

Thus, a worldline is completely characterized by its **flip-count vector** (up to neutral moves). The interval components then encode the kinematics along this worldline.

**11.5.2 Context velocity and rapidity**

We now define **context velocity** and **context rapidity** as parameters derived from interval components along a worldline.

Consider a family of flip words parameterized by a continuous parameter , representing successive segments along a worldline, with corresponding increments:

For a timelike segment (), define the **context velocity** as

which automatically satisfies because of the invariant relation.

Define the associated **context rapidity** by

so that

Here is the **context Lorentz factor** for that segment.

Structurally:

* **Velocity** is the ratio of spatial to coordinate-time components of the flip-count-derived interval.
* **Rapidity** is a logarithmic parameter that linearly adds under composition of collinear boosts (segments of worldline).

**11.5.3 Time dilation from pivot factors**

For a **single inertial CS frame** , consider two worldlines between the same events:

1. A **rest-like** worldline , where spatial displacement vanishes: .
2. A **moving** worldline , with .

Let the total interval components over the full journey be:

* Rest path: .
* Moving path: .

The invariant relation gives:

* For the rest path: , so (up to sign conventions).
* For the moving path:

Assuming both paths connect the same events in , we can set as measured in . Then,

where and .

Thus, **time dilation** is a direct consequence of:

* Using the same CS frame (same ),
* But different flip-count distributions between spatial and proper-time components along different worldlines.

In the AR language:

* **Pivot factors** (via and the hinge normalization) ensure that the interval components are measured consistently across CS frames.
* Different **context rapidities** correspond to different allocations of flip counts between spatial and temporal directions, yielding the usual factor.

**11.5.4 Relativity of simultaneity from non-commuting flips**

Relativity of simultaneity arises from the **non-commutation** of certain flip operations and the fact that CS frames define their own synchronization surfaces.

Consider two CS frames and in relative motion. A sequence of flips that:

* first maps from to ,
* then defines “simultaneous” events in ,
* then maps back to ,

does not, in general, return the same simultaneity structure as one where:

* simultaneity is defined first in ,
* then the whole slice is mapped to .

Formally:

* Let be a simultaneity slice in , defined by a set of carriers with the same .
* Let be the flip word (or family) that maps carriers from to .
* Let be a simultaneity slice in .

Due to non-commutation and the context dependence of , we generally have:

and

where is the inverse mapping (or approximate inverse, in the sense of neutral equivalence).

This misalignment is precisely the **relativity of simultaneity**:

* Each CS frame defines its own synchronization rules via and the operator algebra.
* Non-commutation of flip operators ensures that simultaneity surfaces do not map trivially between moving frames.

The invariant interval remains unchanged, so all observers agree on , but **time coordinates** and associated simultaneity slices differ.

**11.5.5 Twin paradox in ladder picture**

We can now express the **twin paradox** structurally. Two nested-time matter systems share:

* an initial CS frame where they are at rest relative to each other,
* a final CS frame where they meet again after one of them has taken a “travel” worldline.

Let:

* be the net Lorentz factor along ’s worldline (rest-like path),
* be the frame-averaged Lorentz factor along ’s worldline (travel path).

From the flip-count viewpoint:

* ’s worldline corresponds to a sequence of flip words with at each step; its proper time is approximately equal to the CS-A coordinate time .
* ’s worldline includes segments with ; its total proper time is

where for segments with nonzero context velocity.

Thus, in the AR ladder:

* The difference in experienced proper-time is a **simple sum of local time dilation factors** derived from flip counts along each worldline.
* There is no paradox: both worldlines are defined in terms of the same invariant interval and flip algebra, and the **path dependence** of proper time is expected.

The usual resolution — that the traveling twin’s worldline is not globally inertial (involves accelerations or frame changes) — appears here as:

* segments where the system moves between different CS frames,
* segments where certain flips (e.g. trade, sync, framing) are applied differently,
* leading to a different distribution of flip counts and thus a different proper-time accumulation.

**11.5.6 Connection with null-cone pivot identity**

The null-cone pivot identity (Section 11.3) plays a supporting role in SR kinematics:

* The **null cone** defines the structure of causal relations: events that can be related by sequences of null-like flips (with ).
* The **null-shell collapse** with S² kernel ensures that, at the pivot, directional dependence of null propagation disappears after collapse; only the **radial null-time coordinate** matters.

This implies:

1. **Isotropic speed of light**  
   At the pivot, null propagation is isotropic: the effective speed is the same in all directions, because angular structure of null shells is collapsed to a uniform mode.
2. **Frame-independent null structure**  
   Since the null cone is defined by and the collapse kernel enforces isotropy at the pivot, the **shape** of the null cone (and thus the causal structure) is invariant under context flips between CS frames. This matches the invariance of the light cone under Lorentz transformations.
3. **Consistency with interval relation**  
   The null condition is preserved by both the flip algebra and the null-cone collapse, reinforcing that is a **universal invariant** in the AR framework, not a frame-dependent speed.

Thus, the **same pivot** that enforces isotropic inverse-square behavior for scalar fields also enforces isotropic null propagation, tying together gravitational, electromagnetic-like inverse-square behavior, and SR kinematics under a single hinge structure.

**11.5.7 Summary**

In this subsection we reinterpreted basic special-relativistic effects in the ladder-and-pivot language. Worldlines were written as flip sequences between CS frames, with each context flip contributing to the accumulated interval components defined earlier from flip counts and the invariant interval. A **context velocity** and corresponding **rapidity** were introduced as ladder-side parameters controlling how strongly the pivot factors reweight ticks along different worldlines when seen from a given CS.

Time dilation then appeared as a difference in accumulated between two flip sequences that share endpoints but follow different context paths (e.g. one “moving” worldline vs. a “rest” worldline), with the dilation factor derived from the same invariant interval rather than from a background metric. The **relativity of simultaneity** arose from the non-commutation of flip operators associated with different CS frames: reordering flip sequences changed which events are mapped to the same tick index in a given frame, reproducing the usual failure of global simultaneity. The **twin paradox** was expressed as a comparison between two context-flip histories that depart and reunite at the same hinge band; the difference in the number and type of flips, together with their pivot factors, yields a net discrepancy in accumulated proper time in line with the standard resolution.

Finally, we tied these constructions back to the **null-cone and pivot identities** developed earlier: null-cone shells provide the ladder-side representation of lightlike separation, and the pivot identity at ensures that collapse and retarded propagation agree on the hinge band. This guarantees that the invariant interval, null structure, and ladder-side notion of rapidity are mutually consistent.

In the next section (Section 12), we recast these relativistic structures—together with the hinge, pivot kernels, and inverse-square behavior—inside an abstract present-act engine and feasibility geometry. There, cones, dilation, and gravitational effects appear as properties of tick-local feasibility and shell-wise survival fractions, rather than as properties of a prior spacetime manifold.

**12. Present-Act Engine & Feasibility Geometry**

**12.1 Present-Act Engine: Core Contract**

We now specify the **present-act engine** as a concrete realization of the tick-operator algebra and ladder structure defined in previous sections. The engine describes how a single *present act*—a localized update step—selects one successor carrier from a finite set of candidates, using only discrete predicates (gates) and an acceptance rule. In this volume, we treat the engine purely as a mathematical contract; we do not attach any empirical calibration or simulation design to it.

**12.1.1 Local State Space and Neighbor Candidates**

For the present-act engine, we focus on the evolution from tick (k) to tick (k+1), at a fixed context frame (e.g. a chosen CS and a local patch of the ladder). We specify:

* A **local configuration space** (\mathcal{S}\_k) at tick (k), consisting of:
  + the tick carrier (\mathcal{C}\_k),
  + a finite neighborhood of context bands (n \in {n\_{\min},\dots,n\_{\max}}),
  + and any associated graph neighborhood in (G) (a finite set of nodes and edges).
* A **finite set of neighbor candidates**  
  [  
  \mathcal{N}\_k = { \mathcal{C}\_k^{(a)} : a = 1,\dots, N\_k },  
  ]  
  where each (\mathcal{C}\_k^{(a)}) is a candidate carrier describing a possible way the local configuration could be extended or updated from the current tick. These candidates are generated by:
  + applying primitive operators or their combinations to (\mathcal{C}\_{k-1}) and (G),
  + respecting locality in both tick index and context ladder.

The engine’s task, at each act, is to choose a **single** successor carrier (\mathcal{C}\_{k+1}) from (\mathcal{N}\_k) (or from a derived subset), according to a sequence of discrete gates and a deterministic/lexicographic selection rule.

**12.1.2 Feature Alphabet and Equality-Fit Control**

To compare and gate candidate carriers, we define a finite **feature alphabet** (\mathcal{F}). For each candidate (\mathcal{C}\_k^{(a)}), we compute a finite feature vector  
[  
\mathbf{f}^{(a)}*k \in \mathcal{F}^{m},  
]  
for some fixed integer (m), where each component (f^{(a)}*{k,j}) is drawn from a finite set (e.g. discrete labels, bins, or small integer ranges).

The feature extraction map  
[  
\Phi\_{\mathrm{feat}}:\ \mathcal{N}\_k \to \mathcal{F}^m,\quad \mathcal{C}\_k^{(a)} \mapsto \mathbf{f}^{(a)}\_k,  
]  
must satisfy:

1. **Determinism**: for a given (\mathcal{C}*k) and local graph neighborhood, (\Phi*{\mathrm{feat}}) is a deterministic function with no randomness.
2. **Locality**: features depend only on the local neighborhood (tick, bands, graph) considered in this act.
3. **Frame invariance**: features are defined in a way that does not depend on arbitrary labelings within a CS (e.g. they are invariant under permutations of equivalent nodes).

Equality-fit control is defined in terms of these feature vectors:

* A **feature equality relation** (\sim\_{\mathcal{F}}) is fixed (typically exact equality in each component of the feature vector).
* Two candidates (\mathcal{C}\_k^{(a)}, \mathcal{C}\_k^{(b)}) are **feature-equal** if  
  [  
  \mathbf{f}^{(a)}*k \sim*{\mathcal{F}} \mathbf{f}^{(b)}\_k.  
  ]

This relation will be used in gates that require matching features (e.g. contiguity, consistency) and in the acceptance rule when comparing candidates.

**12.1.3 Boolean/Ordinal Gates**

A **gate** is a function that takes a candidate and returns a discrete evaluation, typically:

* a boolean value (pass/fail), or
* an ordinal score drawn from a finite ordered set (e.g. ({0,1,\dots,L})).

We denote a generic gate by  
[  
G\_\alpha : \mathcal{N}*k \to \mathcal{O}*\alpha,  
]  
where (\mathcal{O}\_\alpha) is finite and ordered. Examples of gate roles include:

* **Basic admissibility gates**  
  Check that a candidate respects capacity constraints, IN/ON role rules, and context-ladder locality.
* **Structural gates**  
  Enforce graph-based conditions (e.g. contiguity on boundary graphs, minimal connectivity, no forbidden patterns).
* **Symmetry/consistency gates**  
  Ensure that candidates are consistent with frame properties, invariants, or previously fixed constraints (e.g. invariance under certain group actions).

The gates are applied in a fixed sequence or as a composite mapping:

* First, a subset of gates define **hard constraints** that candidates must satisfy (boolean pass/fail).
* Remaining gates may assign **ordinal scores** used later in the acceptance step.

Critically:

* **All gates are discrete**: they produce boolean or finite ordinal outputs only.
* **No continuous weights, curves, or floating-point thresholds** are used at the control level. Any continuous quantities (e.g. dimension values, pivot factors) may inform which gates are defined, but they do not appear as tunable weights within the gate evaluations.

**12.1.4 Acceptance Rule (Ratio-Lexicographic)**

Once the gate evaluations have been computed for all candidates in (\mathcal{N}\_k), the engine applies an **acceptance rule** to select a single successor carrier. The acceptance rule has two stages:

1. **Feasibility filtering**
   * Discard all candidates that fail any of the hard boolean gates.
   * Let the remaining feasible set be (\mathcal{N}\_k^{\mathrm{feas}} \subseteq \mathcal{N}\_k). If (\mathcal{N}\_k^{\mathrm{feas}} = \varnothing), the act is deemed blocked (no successor) or additional corrective rules may apply; this corner case is not elaborated here.
2. **Ratio-lexicographic selection**
   * For each candidate (a \in \mathcal{N}*k^{\mathrm{feas}}), form a finite vector of* ***ratio-like scores****:  
     [  
     \mathbf{r}^{(a)}k = (r^{(a)}{k,1}, r^{(a)}*{k,2}, \dots, r^{(a)}*{k,M}),  
     ]  
     where each component (r^{(a)}*{k,j}) is a ratio of counts or differences derived from gate evaluations or other discrete quantities (e.g. relative counts of satisfied structural conditions, minimal path lengths in local graphs, etc.), and all such ratios are rational or discrete.
   * Define a lexicographic ordering on these vectors:  
     [  
     \mathbf{r}^{(a)}*k \prec \mathbf{r}^{(b)}k  
     ]  
     if for the smallest index (j) at which they differ, (r^{(a)}{k,j} < r^{(b)}*{k,j}) under the chosen ordering of the ratio components.
   * Select the candidate(s) with lexicographically minimal (\mathbf{r}^{(a)}\_k). If this set contains more than one candidate (a true tie), a separate tie-breaking mechanism (e.g. a uniform random choice restricted to the tie set) may be invoked; this tie-breaking step is outside the deterministic core and is treated as part of a measurement/selection layer, not the algebraic contract.

The important point is:

* The acceptance rule maps **discrete gate outputs** and **ratio-like scores** into a *total ordering* on (\mathcal{N}\_k^{\mathrm{feas}}), and chooses a single candidate **without introducing any continuous weighting or scoring functions** at the control level.

**12.1.5 Typed Budgets as Read-Out Variables**

After a successor carrier (\mathcal{C}\_{k+1}) is chosen, the engine computes **typed budgets** that quantify the increment from tick (k) to (k+1). These are read-out variables derived from the flip-count or gate data and serve as inputs to larger-scale kinematics:

* **Inner (proper) time increment**: (\Delta\tau\_k).
* **Outer (frame) time increment**: (\Delta t\_k).
* **Spatial increment magnitude**: (|\Delta x\_k|).

These increments are related by the invariant interval structure (§2.5):  
[  
\Delta\tau\_k^2 + c^{-2} |\Delta x\_k|^2 = \Delta t\_k^2,  
]  
for some structural constant (c>0). In the present-act engine:

* (\Delta\tau\_k,\ \Delta t\_k,\ |\Delta x\_k|) are **not** arbitrary continuous variables; they are derived from discrete counts and mapping rules (e.g. number of specific flips, gate events, or context transitions).
* Their values are recorded as part of the tick carrier for the next step:  
  [  
  \mathcal{C}*{k+1} \mapsto (k+1,\ h*{k+1}, IN\_{k+1}, ON\_{k+1}, I\_{k+1}, E\_{k+1}, K,\ \Delta\tau\_k,\Delta t\_k,|\Delta x\_k|).  
  ]

Thus, the present-act engine provides a **complete local contract**:

1. Generate finite neighbor candidates.
2. Compute discrete features and gate evaluations.
3. Apply a ratio-lexicographic acceptance rule to pick a unique successor (up to ties).
4. Read out typed budgets consistent with the invariant interval.

All higher-level constructions (relativistic kinematics, feasibility-based gravity, unified field equations) can be built on top of this engine contract, using only these discrete operations and read-out variables, without adding any continuous control parameters or empirical fitting inside the engine itself.

**12.2 Spatial & Temporal Hinges in the Engine**

The present-act engine refines the abstract invariant interval of §2.5 by specifying how **spatial** and **temporal** hinges enter at the level of a single act. In this subsection, we introduce **formal hinge constants** and describe how they constrain the mapping between discrete acts and the read-out budgets ((\Delta\tau,\Delta t,|\Delta x|)). No numerical values or empirical identifications are assigned; the hinges are treated purely as structural parameters.

**12.2.1 Formal Spatial Hinge (\ell\_{\mathrm{H}})**

At the engine level, we introduce a **spatial hinge** (\ell\_{\mathrm{H}}) as a formal positive constant with the dimension of length:

[  
\ell\_{\mathrm{H}} > 0.  
]

Conceptually, (\ell\_{\mathrm{H}}) represents the smallest spatial increment that can appear as a **coherent, addressable unit** in an act’s read-out:

* When the engine produces a nonzero spatial increment (|\Delta x\_k|) for act (k), we require that  
  [  
  |\Delta x\_k| = n\_k ,\ell\_{\mathrm{H}},  
  ]  
  where (n\_k\in \mathbb{N}\_0) is an integer (possibly zero).
* In particular, a one-unit spatial increment corresponds to (|\Delta x\_k| = \ell\_{\mathrm{H}}), and larger displacements arise as integer multiples.

This discretization is compatible with the flip-count description of §2.5: (|\Delta x\_k|) is determined by counts of spatially “separating” elementary moves (e.g. particular kinds of trades, context shifts, or graph steps), and (\ell\_{\mathrm{H}}) provides a **unit** that converts those counts into a length-like quantity.

In this volume, (\ell\_{\mathrm{H}}) is a placeholder for a hinge scale (often identified with an “inner geometric mean” scale in application), but here it remains an unspecified constant.

**12.2.2 Formal Temporal Hinge (T^\*)**

Analogously, we introduce a **temporal hinge** (T^\*) as a formal positive constant with the dimension of time:

[  
T^\* > 0.  
]

The temporal hinge is the minimal nonzero **outer-time span** that can be associated with a single act, when measured in the context frame of the engine:

* When the engine reports an outer-time increment (\Delta t\_k) at act (k), we set  
  [  
  \Delta t\_k = m\_k, T^\*,  
  ]  
  with (m\_k \in \mathbb{N}\_0).
* A single act that fully uses the hinge span corresponds to (\Delta t\_k = T^\*), while larger spans can be thought of as composites of multiple hinge units.

Just as (|\Delta x\_k|) arises from spatially separating flips, (\Delta t\_k) arises from a combination of all flips in the operator word associated with the act, weighted by their “outer-time” contributions. (T^\*) is the unit converting those contributions into the outer-time budget.

Again, in this theoretical spine we do not assign (T^\*) any specific physical value; it is a structural hinge that anchors the relation between discrete acts and the outer-time axis.

**12.2.3 Inner-Time (Proper-Time) Units and Two-Tick Relation**

The inner-time (proper-time) increment (\Delta\tau\_k) is likewise discretized in terms of a fundamental unit (\tau\_0):

[  
\Delta\tau\_k = p\_k,\tau\_0,\quad p\_k \in \mathbb{N}\_0,\quad \tau\_0 > 0.  
]

The unit (\tau\_0) may be derived from the ledger structure (e.g. minimal nonzero change in record consistent with an act), or defined as a fixed conversion from particular patterns in flip counts (e.g. specific combinations of (F) and (S)). Structurally, we require only:

* (\tau\_0) is positive and fixed within the engine.
* (\Delta\tau\_k) arises from discrete counts of flips and gate events, scaled by (\tau\_0).

The **invariant interval** at act (k) is then expressed as:

[  
\Delta\tau\_k^2 + c^{-2}|\Delta x\_k|^2 = \Delta t\_k^2,  
]  
with:

* (|\Delta x\_k| = n\_k,\ell\_{\mathrm{H}}),
* (\Delta t\_k = m\_k,T^\*),
* (\Delta\tau\_k = p\_k,\tau\_0),

and (c>0) a structural conversion factor relating time and space units.

The three constants ((\tau\_0,\ell\_{\mathrm{H}},T^\*)) are not independent once the algebraic structure is fixed and the invariant relation is enforced; e.g., they satisfy:

[  
\tau\_0^2 + c^{-2}\ell\_{\mathrm{H}}^2 = T^{\*2},  
]  
for the “unit act” with (p\_k = n\_k = m\_k = 1), if such an act is admitted. This is a purely **formal identity** in this volume, encoding how the fundamental units of inner-time, outer-time, and space are related at the hinge level of the engine.

**12.2.4 Two-Anchor Construction (Abstract Form)**

Within the engine, the relation between inner and outer units can be expressed abstractly as a **two-anchor** mapping:

* One anchor is the **inner tick**: an act counted in units of (\tau\_0) (proper-time-like).
* The other anchor is the **outer tick**: an act counted in units of (T^\*) (frame time-like).

We can define an integer-valued map  
[  
\chi:\ \mathbb{N}*0 \times \mathbb{N}0 \to \mathbb{Q}+,\quad (p\_k,m\_k) \mapsto \chi(p\_k,m\_k),  
]  
such that (c) and ((\tau\_0,\ell*{\mathrm{H}},T^\*)) are related consistently whenever the same act is viewed from the inner and outer vantage:

* Inner-vantage description: “this act consumes (p\_k) inner ticks of (\tau\_0)” (proper-time).
* Outer-vantage description: “this act spans (m\_k) outer ticks of (T^\*) and covers (n\_k) spatial units (\ell\_{\mathrm{H}})”.

The two-anchor construction then states that for any act (k), the triple ((p\_k,m\_k,n\_k)) must satisfy the invariant interval relation, thereby enforcing a **compatibility constraint** between inner and outer anchoring of time and space.

This abstract formulation is enough for the purposes of the present-act engine in this volume:

* It guarantees that the engine’s typed budgets ((\Delta\tau\_k,\Delta t\_k,|\Delta x\_k|)) are **not arbitrary**, but anchored to formal hinge units ((\tau\_0,\ell\_{\mathrm{H}},T^\*)) consistent with the invariant interval.
* It allows later constructions (relativistic kinematics, feasibility-based gravity, unified field equations) to refer to these units without fixing them numerically.

**12.2.5 Summary**

In summary, the present-act engine incorporates **spatial** and **temporal hinges** as follows:

* A spatial hinge (\ell\_{\mathrm{H}}) sets the fundamental unit of spatial increment per act.
* A temporal hinge (T^\*) sets the fundamental unit of outer-time increment per act.
* A proper-time unit (\tau\_0) sets the fundamental inner-time increment per act.
* All three are tied together by the invariant interval and by the engine’s discrete flip-count structure.

These hinge constants are treated as **formal parameters** in this volume: they appear only in algebraic relations and in the definitions of budget read-outs. Their numerical values, and any connection to physical scales, belong to separate evidence and application documents.

**12.3 Gravity as Feasibility Geometry (abstract)**

We now describe how **gravity** appears at the level of the present-act engine as a property of **feasibility geometry**: the pattern of which local transitions are allowed or suppressed by a structured gate acting on the engine’s candidates. In this picture, gravitational effects do not come from a separate continuous metric field placed on a background; instead, they arise from how a radial, rotation-invariant gate thins or preserves feasible acts as a function of “distance” from a chosen center. This subsection defines the relevant notions in purely discrete, structural terms.

**12.3.1 Feasibility Regions and ParentGate**

Fix a frame (CS) and a distinguished **center** in the underlying graph (e.g. a node or a small region), which we will call the **gravity center**. Around this center, we define **radial shells** indexed by an integer (k \in \mathbb{N}\_0):

* Shell (k) consists of local configurations whose spatial read-out (|\Delta x|) (from the center) satisfies  
  [  
  k,\ell\_{\mathrm{H}} \le |\Delta x| < (k+1),\ell\_{\mathrm{H}},  
  ]  
  where (\ell\_{\mathrm{H}}) is the spatial hinge unit (§6.2).

The present-act engine generates a finite candidate set (\mathcal{N}\_k) for each act, as in §6.1. A **ParentGate** is then an additional gate that acts on these candidates and depends only on:

* the shell index (k),
* local structural features in a rotation-invariant way (e.g. counts of neighbors, local occupancy, etc.),
* and possibly discrete parameters attached to the center.

Formally, we define a family of predicates  
[  
\mathrm{PG}\_k:\ \mathcal{N}\_k \to {0,1},\quad k = 0,1,2,\dots,  
]  
such that:

1. **Radial dependence**  
   For a candidate (\mathcal{C}\_k^{(a)}) with spatial increment in shell (k), (\mathrm{PG}\_k(\mathcal{C}\_k^{(a)})) depends only on shell index (k) and on local, shell-symmetric features (no dependence on angular labels).
2. **Rotation invariance**  
   If two candidates differ only by a rotation around the center (i.e. they occupy different angular positions but the same shell and have otherwise identical local structure), then  
   [  
   \mathrm{PG}\_k(\mathcal{C}\_k^{(a)}) = \mathrm{PG}\_k(\mathcal{C}\_k^{(b)}).  
   ]
3. **Monotone strictness inward**  
   There exists an ordinal “strictness” parameter (s\_k) for the gate in each shell, with the property that  
   [  
   s\_0 \ge s\_1 \ge s\_2 \ge \dots,  
   ]  
   and the gate is at least as strict in inner shells as in outer ones. This ensures that feasibility is not *more* permissive nearer to the center than farther away.

The boolean output of (\mathrm{PG}\_k) is added to the engine’s gate set (as in §6.1.3). A candidate failing (\mathrm{PG}\_k) is discarded; only those passing are considered in the ratio-lexicographic acceptance step.

**12.3.2 Shell-Wise Survival Fractions and Gravity Amplitude**

Given the ParentGate, we can define, **for the engine**, shell-wise **survival fractions** of candidates. For a fixed shell index (k):

* Let (N\_k) be the number of raw candidates in shell (k) before applying (\mathrm{PG}\_k).
* Let (N\_k^{\mathrm{pass}}) be the number that pass (\mathrm{PG}\_k).

We define the **survival fraction** in shell (k) as  
[  
f\_k :=  
\begin{cases}  
\dfrac{N\_k^{\mathrm{pass}}}{N\_k}, & N\_k > 0,\[0.5em]  
0, & N\_k = 0.  
\end{cases}  
]

These fractions ({f\_k}) encode how strongly the ParentGate permits acts to pass at various distances from the center; they are functions only of:

* shell index (k),
* local structure common to candidates in shell (k),
* and any discrete parameters in the gate definition.

We then define a **dimensionless gravity amplitude** (A\_G(k)) as a shell-wise function of the survival fractions. A natural (but not unique) structural choice is:

[  
A\_G(k) := \phi({f\_j}\_{j\le k}),  
]  
where (\phi) is a monotone functional satisfying:

1. (A\_G(0)) encodes the effect of the gate at the center (inner shell).
2. (A\_G(k)) depends on the cumulative thinning or preservation of feasibility up to shell (k).
3. (A\_G(k)) does not depend on angular placements, only on the radial indexing and gate behavior.

For this volume we do not fix (\phi); we only assume that (A\_G(k)) is:

* non-negative,
* and monotonically related (in a broad sense) to the degree of thinning imposed by the ParentGate.

Thus, (A\_G(k)) is a **dimensionless profile** derived entirely from shell-wise feasibility geometry.

**12.3.3 Weak-Field Asymptotics and Inverse-Square Envelope**

In a regime where the ParentGate is “weak” (i.e. survival fractions (f\_k) are close to 1 in all shells) and structural variations are smooth in (k), we can consider an approximate continuum description in terms of a radial variable (r = k,\ell\_{\mathrm{H}}). Under suitable assumptions about how (f\_k) changes with (k), one can obtain an **inverse-square-like envelope** for the variation of (A\_G(k)) with (k):

[  
A\_G(k) \sim 1 + \varepsilon,\frac{1}{k^2}  
\quad\text{for large }k,  
]  
for some small parameter (\varepsilon) capturing the net effect of the gate (the precise coefficient or sign is not fixed here).

Interpreted in terms of a radial potential (\Phi(r)), the discrete shell behavior can be related to:

* A radial profile obeying a discrete Laplacian or difference equation that approximates a (1/r) or (1/r^2) potential in the continuum limit.
* Flux or “influence” distributions across shells that behave like inverse-square laws, consistent with Gauss-law behavior around the hinge band.

Structurally, the key point is that:

* The ParentGate’s shell-wise thinning of feasibility can be encoded in a smooth function (A\_G(r)) of radius (r).
* Under mild assumptions on how (f\_k) decays or deviates from 1, the resulting radial dependence of (A\_G(r)) supports inverse-square-like field behavior in the weak-gate regime.

In this core volume, we do not specify the exact discretization scheme or the detailed form of the asymptotic law; we only state that the shell-wise feasibility geometry naturally yields such law-like radial envelopes in an appropriate continuum limit.

**12.3.4 Relation to the Invariant Interval and Ladder Geometry**

Gravity as feasibility geometry must be consistent with:

* the **invariant interval** ((\Delta\tau^2 + c^{-2}|\Delta x|^2 = \Delta t^2)) of §2.5,
* the **dimension curve** (D(n)) and pivot profile (g(D(n))) of §4.3,
* and the **gravitational sector** of the ladder (§5.3).

At the present-act level:

* The hinge units ((\tau\_0,\ell\_{\mathrm{H}},T^\*)) set the basic relation between inner-time, outer-time, and space increments for a single act (§6.2).
* The ParentGate modifies which spatial increments (|\Delta x| = k,\ell\_{\mathrm{H}}) are feasible at a given act, hence effectively modifying the distribution of allowed paths in flip-count space.

Viewed at the ladder level:

* The pattern of feasible acts across shells feeds into the ladder’s scalar potential (\Phi\_n) and the effective gravitational action (S\_{\mathrm{grav}}) in §5.3.
* The amplitude (A\_G(k)) can therefore be regarded as the **engine-level counterpart** of the ladder’s gravitational coupling: it is a dimensionless quantity that captures how feasibility is radially modulated and, in aggregate, gives rise to the curvature-like behavior encoded in the ladder’s field equations.

Importantly:

* The present-act engine introduces **no continuous gravitational field** as an input; the gravitational effects are entirely encoded in discrete feasibility changes (via ParentGate) plus the invariant interval and ladder structure.
* The ladder’s gravitational sector, in turn, provides a continuum description of those feasibility patterns, yielding effective field equations with inverse-square-like behavior and horizon formation as described in §5.3.

Thus, at a purely structural level, **gravity as feasibility geometry** means:

* Feasibility of local acts is radially structured around centers by a rotation-invariant gate.
* The resulting shell-wise survival fractions define a dimensionless amplitude profile (A\_G(k)).
* In suitable limits, these discrete patterns reproduce relations characteristic of gravitational fields, without introducing a separate metric field in the control layer of the theory.

**12.4 Nested Sphere Fractal Structure & Horizon Template**

We now describe a nested-sphere construction that realizes the hinge and feasibility geometry on an explicit geometric scaffold. The idea is to build a **fractal onion** of 2-sphere shells, each carrying an effective field or kernel, and to characterize **horizon templates** as particular nested-sphere configurations within this structure. As before, everything is treated at a structural level: we specify the maps and relations, but do not attach empirical parameters.

**12.4.1 IFS on (S^2) and Child-Sphere Tiling**

Let (S^2) denote the unit 2-sphere in (\mathbb{R}^3) with its usual metric. We introduce an **Iterated Function System (IFS)** on (S^2),

[  
\mathcal{F}\_{S^2} = {f\_1, f\_2, \dots, f\_M},\quad f\_i : S^2 \to S^2,  
]

where each (f\_i) is a contraction map on (S^2) (with respect to the spherical metric). These maps define a **tiling** of the sphere by child regions:

* For a given shell index (k), we consider a set of “child spheres” or tiles  
  [  
  S^2\_{k,i} := f\_i(S^2\_{k}),\quad i=1,\dots,M,  
  ]  
  where (S^2\_{k}) is the spherical shell at level (k), initially taken to be the entire unit sphere for (k=0).
* The union  
  [  
  S^2\_{k+1} := \bigcup\_{i=1}^M S^2\_{k,i}  
  ]  
  is a contracted, possibly overlapping cover of (S^2\_k). Depending on the choice of maps, these tiles may be disjoint or have controlled overlaps.

This IFS on (S^2) is used to construct a **boundary fractal** on the hinge band. The attractor of (\mathcal{F}\_{S^2}) is a subset of (S^2) that inherits a dimension close to 2 (precisely 2 in the idealized case), matching the hinge dimension (D=2).

**12.4.2 Concave Shell Projector and Nested Onion**

We introduce a **concave shell projector** (\mathcal{P}\_k) that acts on functions or configurations defined on (S^2\_k). For each radial shell index (k), we define:

* A radius-like parameter (r\_k) (e.g. a discrete index mapping to an effective radius),
* A shell region represented as (S^2\_k), which is topologically (S^2) but with an associated radial label.

The concave shell projector (\mathcal{P}\_k) is an operator that:

* Takes a configuration (\Phi\_k : S^2\_k \to \mathbb{R}) (or (\mathbb{C})) and produces a smoothed or coarse-grained configuration on the next inner shell (S^2\_{k-1}). Symbolically,  
  [  
  \mathcal{P}*k : \Phi\_k \mapsto \Phi*{k-1}^{(\mathrm{proj})}.  
  ]
* Is compatible with the IFS structure; for instance, (\Phi\_{k-1}^{(\mathrm{proj})}) can be obtained by combining contributions from child tiles (S^2\_{k,i}):  
  [  
  \Phi\_{k-1}^{(\mathrm{proj})}(x) = \sum\_{i=1}^M w\_{k,i}(x), \Phi\_k\big(f\_i(x)\big),  
  ]  
  with suitable weights (w\_{k,i}(x)) satisfying normalization conditions.

By iteratively applying shell projectors inward, we build a **nested onion** of shells:

[  
\dots \xrightarrow{\mathcal{P}*{k+1}} S^2\_k \xrightarrow{\mathcal{P}k} S^2{k-1} \xrightarrow{\mathcal{P}*{k-1}} \dots \xrightarrow{\mathcal{P}\_1} S^2\_0.  
]

Each shell (S^2\_k) carries a configuration (\Phi\_k), and the collection ({\Phi\_k}) obeys recursion relations imposed by the projectors. The hinge shell (S^2\_0) at (D=2) serves as the central boundary or pivot in this nested structure.

**12.4.3 Concave vs Convex Encounter Rules**

We distinguish between **concave** and **convex** encounter rules when shells interact or when information propagates across shells:

* **Concave encounters**:  
  Operations that move inward, from outer shells to inner shells, applying projectors like (\mathcal{P}\_k). These preserve or compress information towards the hinge, and tend to emphasize boundary-like (2D) features as we approach (S^2\_0).
* **Convex encounters**:  
  Operations that move outward, from inner shells to outer shells, typically extending or “inflating” configurations away from the hinge. These can be modeled by inverse projectors or expansion operators:  
  [  
  \mathcal{E}*k : \Phi*{k-1} \mapsto \Phi\_k^{(\mathrm{exp})},  
  ]  
  with constraints ensuring consistency with the IFS and the ladder’s dimension profile.

We impose structural consistency conditions:

1. **Projector–expander compatibility**
   * For suitable classes of configurations,  
     [  
     \mathcal{P}\_k \circ \mathcal{E}\_k \sim \mathrm{id},  
     ]  
     up to coarse-graining or equivalence relations.
2. **Hinge-centered recursion**
   * Around the hinge shell (S^2\_0), the interplay of concave and convex operations is constrained so that the dimension (D=2) is fixed and preserved; deviations (\delta D) appear only when moving sufficiently far inwards or outwards.

These encounter rules govern how nested shells exchange information and how “field profiles” move across the onion structure, while maintaining the hinge properties and dimension profile.

**12.4.4 Horizon Templates as Nested-Sphere Configurations**

A **horizon template** is a particular configuration of the nested-sphere structure in which:

* Beyond a certain shell index (k = k\_{\mathrm{hor}}), the concave–convex dynamics effectively **stop propagating** information further out (or further in, depending on perspective).
* Concretely, there exists (k\_{\mathrm{hor}}) such that:  
  [  
  \mathcal{P}*{k*{\mathrm{hor}}+1}\circ \mathcal{E}*{k*{\mathrm{hor}}+1}(\Phi\_{k\_{\mathrm{hor}}}) \sim \Phi\_{k\_{\mathrm{hor}}}  
  ]  
  and further shells (\Phi\_{k>k\_{\mathrm{hor}}}) either vanish (become trivial) or remain isomorphic to (\Phi\_{k\_{\mathrm{hor}}}) under allowed equivalences.

In other words, the nested onion has an effective **outer boundary** at shell (k\_{\mathrm{hor}}) beyond which additional shells do not contribute new dynamical degrees of freedom relevant to the region of interest. This can be seen as:

* A discrete analog of a horizon: from the viewpoint of inner shells, the outer onion beyond (k\_{\mathrm{hor}}) is either invisible or indistinguishable (up to equivalence).

Formally, we can define:

* A **horizon shell index** (k\_{\mathrm{hor}}),
* A **horizon template** ({\Phi\_k}*{k\le k*{\mathrm{hor}}}) that satisfies the above recursion and fixed-point properties,
* and boundary conditions for outer shells (e.g. (\Phi\_k = 0) or (\Phi\_k \sim \Phi\_{k\_{\mathrm{hor}}}) for all (k>k\_{\mathrm{hor}})).

The nested-sphere IFS and shell projectors allow such templates to be constructed in a way that is compatible with the hinge and with the ladder’s gravitational sector (§5.3).

**12.4.5 Structural Role in Feasibility-Based Gravity**

In the context of feasibility-based gravity (§6.3), the nested-sphere structure plays several roles:

1. **Geometric realization of shells**
   * The radial shells defined in terms of (|\Delta x|) and ParentGate can be associated with specific shells (S^2\_k) in the nested onion.
2. **Boundary conditions for feasibility**
   * Horizon templates provide a structural origin for “no further outward propagation” conditions, matching the idea that beyond some radial shell, feasible acts that connect to interior configurations are effectively blocked.
3. **Coupling to ladder-level potentials**
   * The scalar potential (\Phi\_n) in §5.3 can be realized as a nested-sphere field (\Phi\_k) on each shell, with the hinge shell (S^2\_0) serving as the pivot where the nested onion meets the ladder’s hinge band.

By combining:

* the discrete, shell-based feasibility geometry (via ParentGate and survival fractions), and
* the nested-sphere fractal onion (via (\mathcal{F}\_{S^2}) and shell projectors),

the theory achieves a **geometric template** for horizons and gravitational behavior that remains fully discrete at the engine level, yet readily admits continuum interpretations at the hinge and ladder scales.

No empirical parameter choices enter this construction; the nested-sphere structure is a purely theoretical scaffold enabling the definition of horizons, boundary conditions, and gravitational templates consistent with the rest of the V1 framework.